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by

L. J. PASCOE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SYNTHETIC SEISMOGRAMS BY THE ALEKSEEV-MIKHAILENKO METHOD submitted by L. J. PASCOE in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE.

Abstract

The production of synthetic seismograms using a technique originally developed by Alekseev and Mikhailenko is presented. An explicit finite-difference solution to the elastodynamic wave equation for a vertically inhomogeneous medium is determined after reducing the dimensionality of the wave equation through the use of finite Hankel transforms. The finite difference scheme used for both SH and P-SV cases is outlined in detail as well as the stability criterion for each. Seismic sources considered include SH point torque source, horizontal point force, vertical point force, and explosive. The numerical models presented illustrate the nature of wave propagation for a particular model containing two thin low-velocity layers in a homogeneous half-space.

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1. INTRODUCTION

The computation of synthetic seismograms using a finite-difference method is considered for the case of a vertically inhomogeneous medium. Finite Hankel transforms are used to reduce the dimensionality of the wave equation and allow the use of an explicit finite-difference scheme.

1.1 Equations of Motion

For a vertically inhomogeneous medium the equation describing the motion is given by Grant and West (1965):

1.1 $$\rho\ddot{\vec{u}} = (\lambda+\mu)\nabla\theta + \theta\nabla\lambda + \mu\nabla^2\vec{u} + (\nabla\mu\cdot\nabla)\vec{u} + \nabla(\nabla\mu\cdot\vec{u}).$$

In the following we use cylindrical coordinates (r,ϕ,z) with the z-axis pointing downward into the vertically inhomogeneous half-space whose surface is at $z=0$. The density is given by $\rho(z)$ and Lamé's parameters are given by $\lambda(z)$ and $\mu(z)$. Using $\theta=\nabla\cdot\vec{u}$ and the vector definition in cylindrical coordinates $\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times\nabla\times\vec{u}$ this reduces to

1.2 $$\rho\ddot{\vec{u}} = (\lambda+2\mu)\nabla(\nabla\cdot\vec{u}) + (\nabla\cdot\vec{u})\nabla\lambda - \mu(\nabla\times\nabla\times\vec{u})$$
$$+ (\nabla\mu\cdot\nabla)\vec{u} + \nabla(\nabla\mu\cdot\vec{u})$$

where $\vec{u} = (u_r, u_\phi, u_z)$ and
$$u_r = u_r(r,z,t)$$
$$u_\phi = u_\phi(r,z,t)$$
$$u_z = u_z(r,z,t)\,.$$

To illustrate the decoupling of the SH and P-SV types we may consider the equation of motion given by 1.2 in cylindrical coordinates:

$$\nabla(\nabla\cdot\vec{u}) = \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right)\right\}\hat{r}$$

$$+ \left\{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right)\right\}\hat{z}$$

$$\nabla\cdot\vec{u}\nabla\lambda = \left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right\}\frac{\partial\lambda}{\partial z}\quad\hat{z}$$

$$\nabla\times\nabla\times\vec{u} = -\frac{\partial}{\partial z}\left\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right\}\hat{r} - \left\{\frac{\partial^2 u_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\phi)\right)\right)\right\}\hat{\phi}$$

$$+ \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial z}\right)\right)\right\}\hat{z}$$

$$(\nabla\mu\cdot\nabla)\vec{u} = \frac{\partial\mu}{\partial z}\frac{\partial}{\partial z}(u_r\hat{r} + u_\phi\hat{\phi} + u_z\hat{z})$$

$$= \left(\frac{\partial\mu}{\partial z}\frac{\partial u_r}{\partial z}\right)\hat{r} + \left(\frac{\partial\mu}{\partial z}\frac{\partial u_\phi}{\partial z}\right)\hat{\phi} + \left(\frac{\partial\mu}{\partial z}\frac{\partial u_z}{\partial z}\right)\hat{z}$$

$$\nabla\mu\cdot\vec{u} = \left(\frac{\partial\mu}{\partial z}\hat{z}\right)\cdot(u_r\hat{r} + u_\phi\hat{\phi} + u_z\hat{z})$$

$$= \frac{\partial\mu}{\partial z}u_z$$

$$\nabla(\nabla\mu\cdot\vec{u}) = \frac{\partial}{\partial r}\left(\frac{\partial\mu}{\partial z}u_z\right)\hat{r} + \frac{\partial}{\partial z}\left(\frac{\partial\mu}{\partial z}u_z\right)\hat{z}$$

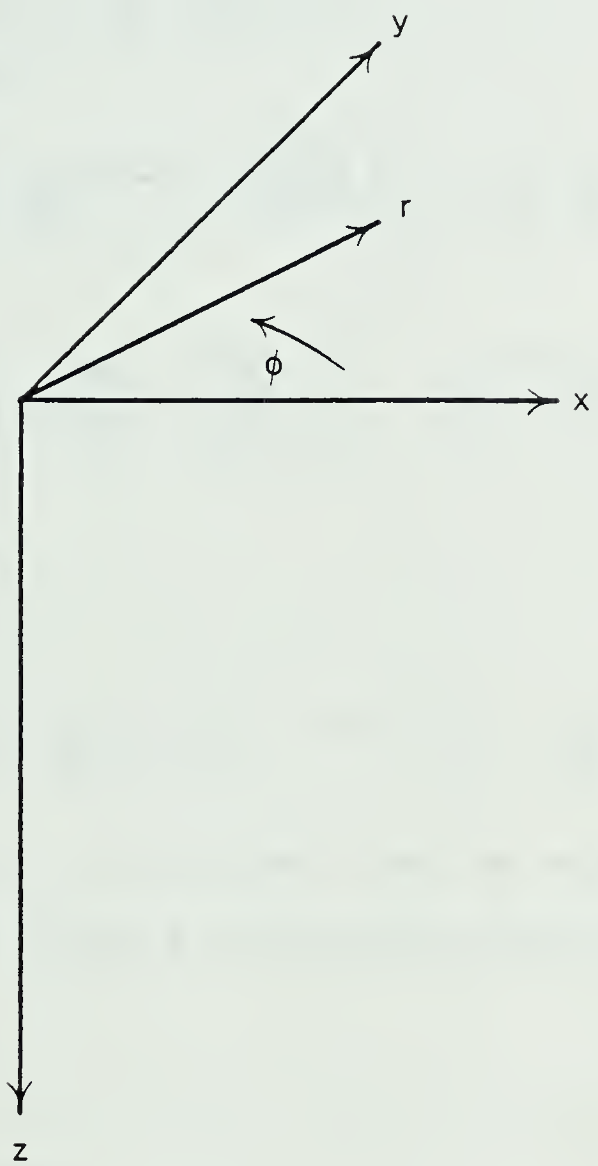


Figure 1.1 Coordinate axes orientation.

The equations of motion become

Radial Component:

$$\rho \frac{\partial^2 u_r}{\partial t^2} = \eta \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} \right) \right\} + \mu \frac{\partial}{\partial z} \left\{ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right\}$$

1.3

$$+ \left(\frac{\partial \mu}{\partial z} \frac{\partial u_r}{\partial z} \right) + \frac{\partial}{\partial r} \left(\frac{\partial \mu}{\partial z} u_z \right)$$

Vertical Component:

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \eta \left\{ \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} \right) \right\} + \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} \right\} \frac{\partial \lambda}{\partial z}$$

1.4

$$- \frac{\mu}{r} \left\{ \frac{\partial}{\partial r} \left(r \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial z} \right) \right) \right\} + \frac{\partial \mu}{\partial z} \frac{\partial u_z}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\partial \mu}{\partial z} u_z \right)$$

Azimuthal Component:

1.5

$$\rho \frac{\partial^2 u_\phi}{\partial t^2} = \mu \left\{ \frac{\partial^2 u_\phi}{\partial z^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\phi) \right) \right) \right\} + \frac{\partial \mu}{\partial z} \frac{\partial u_\phi}{\partial z}$$

In this form it is evident that SH waves, given by 1.5 separate from the coupled P-SV waves described by equations 1.3 and 1.4.

1.2 Seismic Sources

Body force terms representing seismic sources may also be written in terms of potentials to incorporate in the equations of motion. For SH waves we may consider the case of a torque source at the free surface, z=0. The body force is defined in terms of the potential ν as (Aki and Richards (1980) Vol I):

1.6 $$\vec{F} = \nabla\times(0,0,\nu).$$

Both Cartesian coordinates (x,y,z) and cylindrical coordinates (r,ϕ,z) can be used where they share the same depth dependence:

1.7 $$\nu(x,y,z) = \delta(x)\delta(y)\delta(z)f(t)$$

$$\nu(r,\phi,z) = \frac{\delta(r)\delta(z)}{2\pi r}\; f(t)$$

where δ represents the Dirac delta function and f(t) represents the time dependence of the source. As a mathematical model, the point torque source is thus expressed in terms of this potential as

1.8 $$\begin{aligned}\vec{F}(r,z,t) &= \nabla\times(0,0,\nu)\\ &= -\frac{\partial\nu}{\partial r}\hat{\phi}\\ &= -\frac{f(t)\delta(z)}{2\pi}\frac{\partial}{\partial r}\left(\frac{\delta(r)}{r}\right)\hat{\phi}\;.\end{aligned}$$

For an explosive source, the body force term is defined as

1.9 $$\begin{aligned}\vec{F}(r,z,t) &= \nabla\nu\\ &= f(t)\;\nabla\left\{\frac{\delta(r)\delta(z-d)}{2\pi r}\right\}\end{aligned}$$

where the source is assumed to be at depth z=d.

In the case of a vertical point force, the force may be expressed directly, rather than in terms of a potential as

$$\vec{F}(r,z,t) = \frac{f(t)\delta(r)\delta(z-d)}{2\pi r}\,\hat{z}\ . \tag{1.10}$$

2. SH CASE

2.1 Statement of the Problem

For the case of a SH torque source, only the u component is nonzero due to the decoupling of P-SV from SH motion for isotropic media. The equation for $u = u_\phi(r,z,t)$ becomes

2.1

$$\frac{\rho(z)}{\mu(z)} \frac{\partial^2 u}{\partial t^2} = \frac{\mu'(z)}{\mu(z)} \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2}$$

subject to initial conditions

2.2

$$u \bigg|_{t=0} = \frac{\partial u}{\partial t}\bigg|_{t=0} = 0.$$

The boundary condition imposed by a torque source at the free surface z=0 imposes the boundary condition

2.3

$$\tau_{\phi z}\bigg|_{z=0} = - \frac{f(t)}{2\pi} \frac{d}{dr}\left(\frac{\delta(r)}{r}\right)$$

from which is obtained

2.4

$$\frac{\partial u}{\partial z}\bigg|_{z=0} = - \frac{f(t)}{2\pi\mu_o} \frac{d}{dr}\left(\frac{\delta(r)}{r}\right).$$

The statement of the problem for the case in which only SH waves appear is given by equation 2.1 subject to initial conditions 2.2 and boundary condition 2.4. The

dimensionality of the equation of motion can be reduced by the application of a first order finite Hankel transform.

The determination of the kernel to be of value in transforming the equation is obtained by considering solutions of the equation Lu=0 where L denotes the linear differential operator

2.5
$$Lu = \frac{1}{p(r)}\left\{\frac{\partial}{\partial r}\, q(r)\, \frac{\partial u}{\partial r}\right\} + m(r)u.$$

The equation of motion for SH waves can be expressed in terms of this operator as

2.6
$$\frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)\right\} - \frac{\nu^2}{r^2}\,u + \frac{\partial^2 u}{\partial z^2} + \frac{1}{\mu(z)}\,\frac{\partial \mu}{\partial z}\,\frac{\partial u}{\partial z} - \frac{\rho(z)}{\mu(z)}\,\frac{\partial^2 u}{\partial t^2} = 0.$$

The finite transforms corresponding to the operator

2.7
$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} - \frac{\nu^2}{r^2}$$

are called finite Hankel transforms. If we consider the solution of the Sturm-Liouville equation

$$(L-\xi)u = 0$$

where we take $\xi=-k_i^2$ then the equation reduces to Bessel's equation:

2.8
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \left(k_i^2 - \frac{\nu^2}{r^2}\right) = 0.$$

Imposing the boundary condition $u|_{r=a}=0$ it has the general solution $u(r)=J\nu(k_i r)$ where k_i are the roots of $J\nu(k_i a)=0$. (Sneddon (1972)).

The transform to use in the case of SH waves, for $\nu=1$, is a finite first order Hankel transform of the first kind

2.9

$$S(k_i,z,t) = \int_0^a u(r,z,t)J_1(k_i r)rdr.$$

The inversion formula is given by

2.10

$$u(r,z,t) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{S(k_i,z,t)J_1(k_i r)}{(J_2(k_i a))^2} .$$

The application of the transform is simplified by writing equation 2.1 in modified form

2.11

$$u(z)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)=\rho(z)\frac{\partial^2 u}{\partial t^2} .$$

For the first term, integration twice by parts yields the following result:

$$\int_0^a u(z)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)J_1(k_i r)rdr$$

$$= -\mu\left\{ak_i J_o(k_i a)u\Big|_{r=a} - k_i^2\, S(k_i,z,t)\right\}.$$

If the boundary condition $u|_{r=a}=0$ is imposed, it is equivalent to the introduction of a totally reflecting

surface at r=a. If this reflecting surface is set far enough from the source, the seismic response in the region of interest will not be influenced by any spurious reflections. The equation of motion upon application of the transform becomes

2.12
$$\frac{\partial}{\partial z}\left(\mu(z)\,\frac{\partial S}{\partial z}\right) - k_i^2\mu(z)S = \rho(z)\,\frac{\partial^2 S}{\partial t^2} \qquad \begin{array}{l} t\geq 0 \\ z\geq 0. \end{array}$$

The boundary and initial conditions given by equations 2.2 and 2.4 are transformed to yield

2.13
$$S\Big|_{t=0} = \frac{\partial S}{\partial t}\Big|_{t=0} = 0$$

2.14
$$\frac{\partial S}{\partial z}\Big|_{z=0} = -\,\frac{f(t)}{2\pi\mu_o}\int_0^a \frac{d}{dr}\left(\frac{\delta(r)}{r}\right)J_1(k_i r)\,r\,dr$$
$$= \frac{f(t)}{2\pi\mu_o}\int_0^a \frac{\delta(r)}{r}\,k_i\,J_o(k_i r)\,r\,dr$$
$$= \frac{f(t)k_i}{4\pi\mu_o}\,.$$

In addition to the reflections from the fictitious surface at r=a, since the depth is finite when the problem is solved by numerical methods, there is a need to minimize reflections from the lower grid boundary. This may be accomplished by having the depth great enough that reflections do not influence the seismic record for the time being considered. Another method which minimizes such

reflections and results in a manageable grid depth requires the introduction of a damping term $\gamma(z)\ \partial u/\partial t$ in the equation of motion. In effect reflections remain but, because they are heavily damped, their amplitude will be negligible. The equation of motion becomes

2.15
$$\frac{\partial}{\partial z}\left(\mu(z)\ \frac{\partial S}{\partial z}\right) - k_i^2\mu(z)S = \rho(z)\ \frac{\partial^2 S}{\partial t^2} + \gamma(z)\ \frac{\partial S}{\partial t}$$

where $\gamma(z)=0$ in the region of interest and increases linearly with depth over several wavelengths to the lower grid boundary.

2.2 Finite-Difference Equation

The set of equations to be solved by finite-difference methods is given by

2.15
$$\frac{\partial}{\partial z}\left(\mu(z)\ \frac{\partial S}{\partial z}\right) - k_i^2\mu(z)S = \rho(z)\ \frac{\partial^2 S}{\partial t^2} + \gamma(z)\ \frac{\partial S}{\partial t}$$

2.13
$$S\Big|_{t=0} = \frac{\partial S}{\partial t}\Big|_{t=0} = 0$$

2.14
$$\frac{\partial S}{\partial z}\Big|_{z=0} = \frac{k_i f(t)}{4\pi\mu_o}$$

where k_i is the i-th root of $J_1(k_i a)=0$. For a given root, this system can be solved efficiently by the use of an explicit finite-difference method. It is necessary to limit

grid dispersion by the selection of a sufficiently fine spatial grid. The grid dispersion causes a delaying and broadening of the signal such that the pulses develop an oscillatory tail. In the programs used it was necessary to use a minimum of 40 grid steps per wavelength.

The determination of a finite-difference analogue for equation 2.15 is facilitated through the use of a relation given by Mitchell (1977):

2.16

$$\frac{\partial}{\partial z}\left(\mu(z)\ \frac{\partial S}{\partial z}\right)\Bigg|_{z=z_k} = \frac{a_k S^i_{k-1} - (a_k + a_{k+1}) S^i_k + a_{k+1} S^i_{k+1}}{(\Delta z)^2} + 0(\Delta z)^2$$

where

2.17

$$a_k = \Delta z \left\{ \int_{z_{k-1}}^{z_k} \frac{dz}{\mu(z)} \right\}^{-1}$$

$$= \frac{2\mu_k \mu_{k-1}}{\mu_k + \mu_{k-1}} .$$

In this relation the i-th time step has been considered and the finite-difference formula is centered about depth $z=z_k$.

The formal development of this relationship is as follows:

$$\frac{\omega}{\mu(z)} = \frac{\partial S}{\partial z}$$

$$\omega_{k-\frac{1}{2}} \int_{z_{k-1}}^{z_k} \frac{dz}{\mu(z)} = S_k - S_{k-1}$$

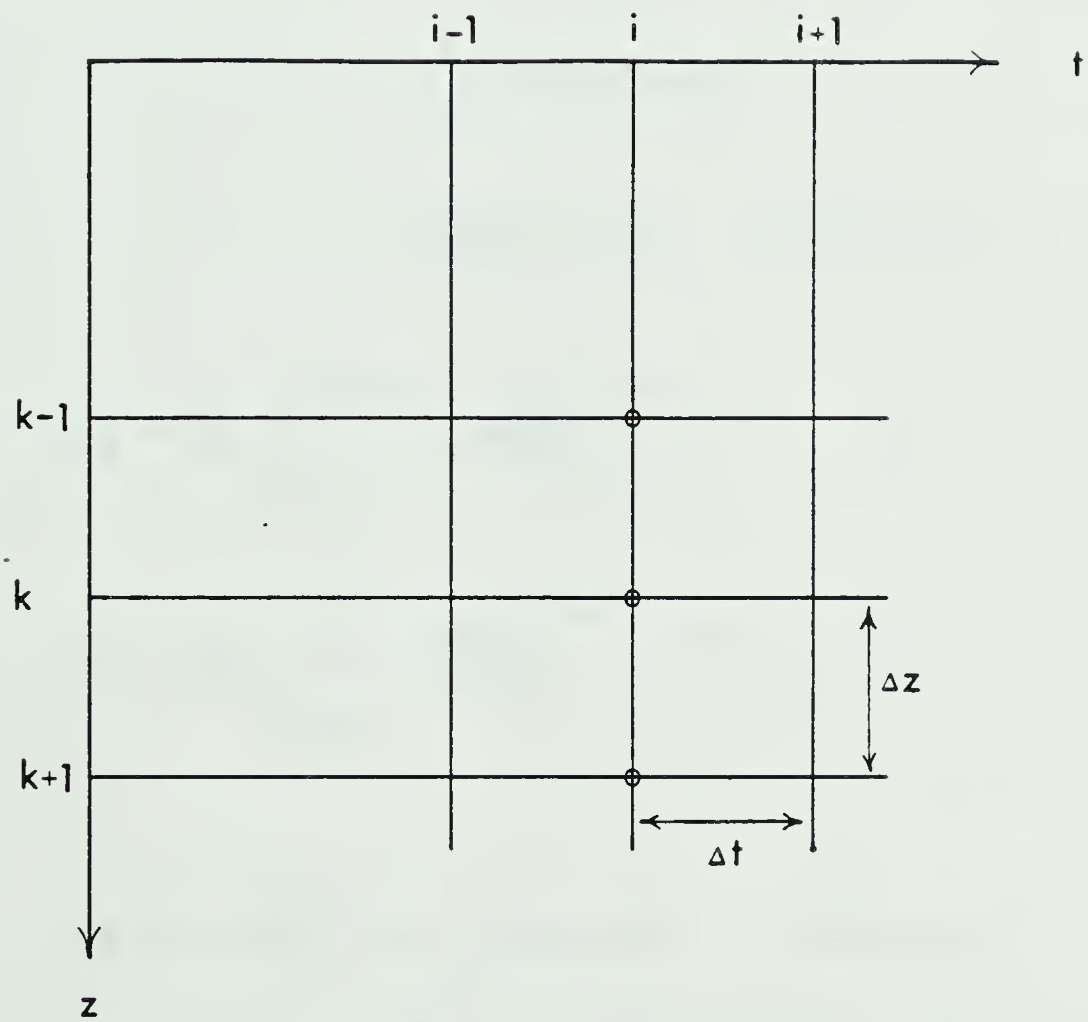


Figure 2.1 Finite-difference grid.

$$\omega_{k+\frac{1}{2}} \int_{z_k}^{z_{k+1}} \frac{dz}{\mu(z)} = S_{k+1} - S_k$$

$$\therefore \qquad \frac{\partial}{\partial z}\left(\mu(z)\ \frac{\partial S}{\partial z}\right) = \frac{\Delta\omega}{\Delta z}$$

$$= \frac{1}{\Delta z}\ (\omega_{k+\frac{1}{2}} - \omega_{k-\frac{1}{2}})$$

$$= B_k(S_{k+1} - S_k) - A_k(S_k - S_{k-1})$$

where
$$B_k = \frac{1}{\Delta z}\left\{\int_{z_k}^{z_{k+1}} \frac{dz}{\mu(z)}\right\}^{-1}$$

$$A_k = \frac{1}{\Delta z}\left\{\int_{z_{k-1}}^{z_k} \frac{dz}{\mu(z)}\right\}^{-1}$$

hence

$$\frac{\partial}{\partial z}\left(\mu(z)\ \frac{\partial S}{\partial z}\right) = A_{k+1}(S_{k+1} - S_k) - A_k(S_k - S_{k-1})$$

$$= \frac{a_k S_{k-1} - (a_k + a_{k-1}) S_k + a_{k+1} S_{k+1}}{(\Delta z)^2} .$$

The finite-difference analogue to 2.15 is given by

2.18

$$\frac{a_k S_{k-1}^{i} - (a_k + a_{k+1}) S_k^{i} + a_{k+1} S_{k+1}^{i}}{(\Delta z)^2} - k_i^2 \mu_k S_k^{i-1}$$

$$= \rho_k \left\{ \frac{S_k^{i+1} - 2S_k^{i} + S_k^{i-1}}{(\Delta t)^2} \right\} + \gamma_k \left\{ \frac{S_k^{i+1} - S_k^{i-1}}{2\Delta t} \right\}.$$

In this form it is clear that S_k^{i+1} can be solved for explicitly at time step (i+1) in terms of known values of S_k at the previous time steps i and (i-1).

2.19

$$\left\{ \frac{\rho_k}{(\Delta t)^2} + \frac{\gamma_k}{2\Delta t} \right\} S_k^{i+1} = \frac{a_k S_{k-1}^{i}}{(\Delta z)^2} - \left\{ \frac{(a_k + a_{k+1})}{(\Delta z)^2} - \frac{2\rho_k}{(\Delta t)^2} + k_i^2 \mu_k \right\} S_k^{i}$$

$$+ \frac{a_{k+1} S_{k+1}^{i}}{(\Delta z)^2} - \left\{ \frac{\rho_k}{(\Delta t)^2} - \frac{\gamma_k}{2\Delta t} \right\} S_k^{i-1}$$

$$1 \le k \le m$$

2.3 Free Surface Condition

At the free surface, k=0, direct application of 2.19 results in the introduction of a fictitious grid point element S_{-1}^{i}. To overcome this difficulty it can be assumed that the medium is homogeneous for k=0,1,2 and that no damping term is required. With these restrictions equation 2.15 can be written in simplified form:

2.20

$$\mu(z) \frac{\partial^2 S}{\partial z^2} - k_i^2 \mu(z) S = \rho(z) \frac{\partial^2 S}{\partial t^2} .$$

As a finite-difference equation this becomes

2.21 $$\left(\frac{\partial^2 S}{\partial z^2}\right)_o^i - k_i^2\, S_o^i = \left\{\frac{1}{v_s^2}\left(\frac{\partial^2 S}{\partial t^2}\right)\right\}_o^i \qquad \text{where } v_s^2 = \frac{\mu(z)}{\rho(z)}$$

$$\frac{S_{-1}^i - 2S_o^i + S_1^i}{(\Delta z)^2} - k_i^2\, S_o^i = \frac{1}{v_s^2}\left\{\frac{S_o^{i+1} - 2S_o^i + S_o^{i-1}}{(\Delta t)^2}\right\}.$$

To remove the fictitious value S_{-1}^i boundary condition 2.14 may be used. Its finite-difference equivalent is

2.22 $$\frac{S_1^i - S_{-1}^i}{2\Delta z} = \frac{f(t)k_i}{4\pi\mu_o}.$$

Substituting for S_{-1}^i in the equation of motion at the free surface, the explicit solution for S_o^{i+1} is given by

2.23 $$S_o^{i+1} = \left(2v_s^2\,\frac{(\Delta t)^2}{(\Delta z)^2}\right) S_o^i - \left\{\frac{2v_s^2(\Delta t)^2}{(\Delta z)^2} + k_i^2 v_s^2 (\Delta t)^2 - 2\right\} S_o^i$$

$$-S_o^{i-1} - \frac{k_i f(t)(\Delta t)^2}{2\pi\rho_o(\Delta z)}.$$

2.4 Seismic Pulse

In the computer programs the form of the seismic pulse f(t) was that of an exponentially damped sine function expressed as

2.24 $$f(t) = \sin(2\pi f_o t)\,\exp\left\{-\left(\frac{2\pi f_o t}{\sigma}\right)^2\right\}.$$

where f_o is the predominant frequency and the factor σ

controls the damping.

In order to determine the seismic response it is necessary to evaluate $S(k_i,z,t)$ for each root k_i over the z-t grid. The response at any given depth z_0 and epicentral distance r_0 is found from the inverse transform

2.25
$$u(r_o,z_o,t) = \frac{2}{a^2}\sum_{i=1}^{\infty}\frac{S(k_i,z,t)J_1(k_i r_o)}{(J_o(k_i a))^2}$$

where the relation $J_2(k_i a) = J_1'(k_i a) = J_0(k_i a)$ has been used (Abramowitz and Stegun (1968)). In solving for $S(k_i,z,t)$, equations 2.19 and 2.23 are used in conjunction with the initial conditions imposed by 2.13. The error introduced by truncation of the series is of the order 2-3% over 100λ provided a sufficient number of roots k_i are considered. For the case f_0=1 and σ=4, a good estimate for the number of terms in equation 2.25 is "4a". For the P-SV case, extensive investigation of the technique places the number of terms at "9a" (Personal communication B. G. Mikhailenko).

2.5 Numerical Considerations

The number of calculations required can be reduced by a consideration of the rate at which the disturbance propagates through the medium. The finite-difference equations are to be solved for all depths $z=k(\Delta z)$, $0 \le k \le m$, at a given time step. The points in z to which the disturbance has not yet propagated may be omitted from the calculation in the initial portion of the grid since $S(k_i,z,t) \equiv 0$ in that

region.

In addition, for a given receiver depth z_0, it is unnecessary to compute $S(k_i,z,t)$ at those grid points that will influence the disturbance after T, the length of the seismogram. A consideration of the grid elements used in the finite-difference calculation, Figure 2.2, shows that a unit step staircase can be used from the lower grid boundary to the depth z_0 in excluding grid points from the calculation in the latter portion of the grid as illustrated in Figure 2.3.

2.6 Stability

Because there is a possibilty of unlimited amplification of errors by the finite-difference method for arbitrary choice of Δz and Δt, the stability criteria for the technique must be determined. The von Neumann condition for stability can be applied because of the separability of variables in the problem. In this method a harmonic decomposition of the error E is made at grid points at a given time level. The form of the error function E can be written as (Mitchell (1977)):

2.26

$$E = \sum_j A_j e^{i\beta_j \Delta z}$$

$$= \sum_j A_j e^{i\beta_j (k\Delta z)}$$

where, in general, the frequencies β_j are arbitrary. It is

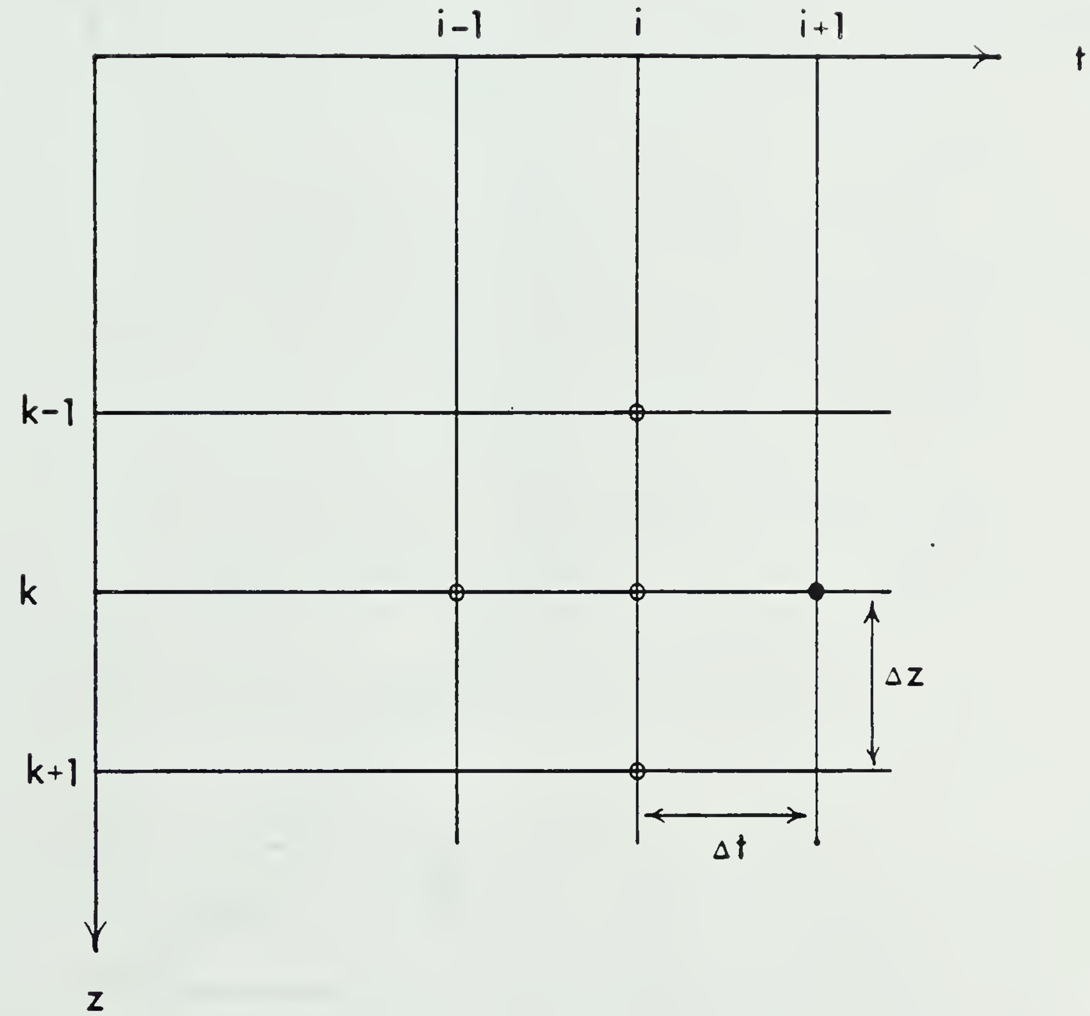


Figure 2.2 Grid points used in the finite-difference calculation.

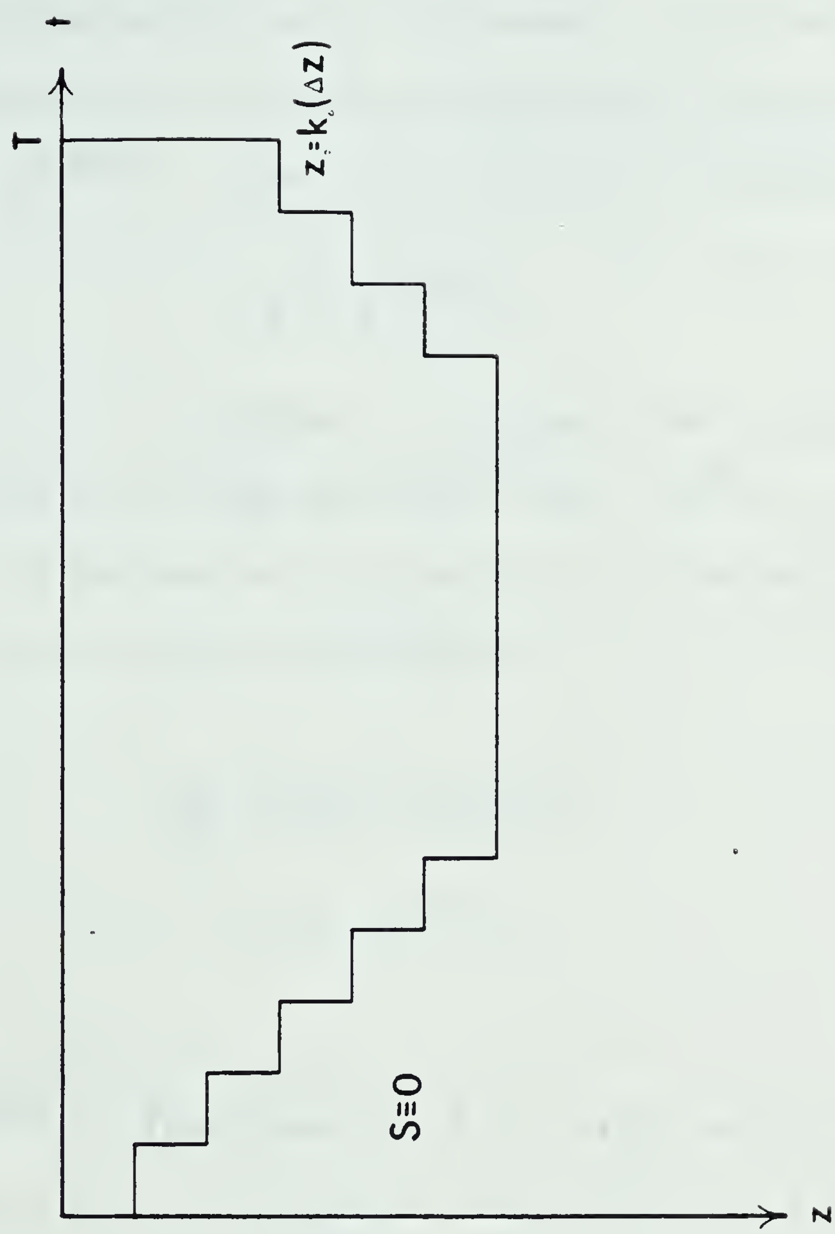


Figure 2.3 Grid points used to calculate the response.

necessary to consider only the single term $e^{i\beta k\Delta z}$ where β is any real number. For convenience, suppose that the time level being considered corresponds to t=0. To investigate the error propagation as t increases, it is necessary to find a solution of the finite-difference equation which reduces to $e^{i\beta k\Delta z}$ when t=0. Let such a solution be

$$e^{\alpha t}\, e^{i\beta k\Delta z}$$

where $\alpha=\alpha(\beta)$ is, in general, complex. The original error will not grow with time provided that $|e^{\alpha\Delta t}|<1$ is satisfied for all α. To determine if the error grows with time, let the error be written in the form

2.27

$$E_k^n = e^{\alpha n\Delta t}\, e^{i\beta k\Delta z}$$

$$= \lambda^n\, e^{i\beta k\Delta z}$$

where $\lambda=e^{\alpha\Delta t}$. The quantity λ is referred to as the amplification factor of the finite-difference equation. To facilitate substitution of the error function into equation 2.19 the variable coefficients are frozen to their values at a specific grid point and the stability condition determined from the von Neumann criterion by testing the modified equation with constant coefficients. The test should in theory be carried out for all points in the region under consideration.

Writing $v_s^2 = \mu/\rho$ and $a=\mu$ equation 2.19 can be written as

2.28
$$S_k^{i+1} = v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 S_{k-1}^i - \left\{2v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 - 2 + k_i^2 v_s^2 (\Delta t)^2\right\} S_k^i$$

$$+ v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 S_{k+1}^i - S_k^{i-1} .$$

Substitution of the error function into this equation yields

2.29

$$\lambda^{n+1} e^{i\beta k\Delta z} = v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \lambda^n e^{i\beta(k-1)\Delta z}$$

$$+ \left\{2 - 2v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 - k_i^2 v_s^2 (\Delta t)^2\right\} \lambda^n e^{i\beta k\Delta z}$$

$$+ v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \lambda^n e^{i\beta(k+1)\Delta z} - \lambda^{n-1} e^{i\beta k\Delta z}$$

2.30

$$\lambda + \frac{1}{\lambda} = v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \left\{e^{i\beta\Delta z} + e^{-i\beta\Delta z}\right\}$$

$$+ \left\{2 - 2v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 - k_i^2 v_s^2 (\Delta t)^2\right\}$$

$$= v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \left\{2\cos\beta\Delta z - 2\right\} + 2 - k_i^2 v_s^2 (\Delta t)^2$$

$$= v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \left\{-4\sin^2 \frac{\beta\Delta z}{2}\right\} + 2 - k_i^2 v_s^2 (\Delta t)^2$$

2.31

$$\lambda^2 - \left\{v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \left\{-4\sin^2 \frac{\beta\Delta z}{2}\right\} + 2 - k_k^2 v_s^2 (\Delta t)^2\right\} \lambda + 1 = 0.$$

Equation 2.31 may be written in the form

$$\lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

where λ_1 and λ_2 are the roots. For stability it is necessary that $|\lambda_1|<1$ and $|\lambda_2|<1$, hence

$$\begin{aligned} |\lambda_1 + \lambda_2| &\le |\lambda_1| + |\lambda_2| \\ &< 2 \end{aligned}$$

Therefore

$$\left| v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 \left\{- 4\sin^2 \frac{\beta\Delta z}{2}\right\} + 2 - k_i^2 v_s^2 \ (\Delta t)^2 \right| < 2$$

$$\left| 2 - 4\ v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 - k_i^2\ v_s^2 (\Delta t)^2 \right| < 2$$

The corresponding inequalities are

2.32

$$v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 + \frac{k_i^2}{4}\ v_s^2 (\Delta t)^2 > 0$$

2.33

$$v_s^2\left(\frac{\Delta t}{\Delta z}\right)^2 + \frac{k_i^2}{4}\ v_s^2 (\Delta t)^2 < 1.$$

The stability condition is given by the last inequality since the first one is trivially satisfied.

3. P-SV CASE

3.1 Equations of Motion

The equation of motion for a vertically inhomogeneous medium is given by

3.1

$$\rho\ddot{\vec{u}} = (\lambda+2\mu)\nabla(\nabla\cdot\vec{u}) + (\nabla\cdot\vec{u})\nabla\lambda - \mu(\nabla\times\nabla\times\vec{u})$$

$$+ (\nabla\mu\cdot\nabla)\vec{u} + \nabla(\nabla\mu\cdot\vec{u}).$$

Assuming that Lamé's parameters are functions of depth only $\mu=\mu(z)$, $\lambda=\lambda(z)$, and $(\lambda+2\mu)=\eta(z)$ the terms of equation 3.1 have the following forms in cylindrical coordinates where $u = (u_r, u_\phi, u_z)$. The radial and vertical components of the displacement vector $\vec{u}$ are given by u_r and u_z respectively and may be expressed as

Radial Component:

3.2 $$\eta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial}{\partial r}\left(\eta \frac{\partial u_z}{\partial z}\right) - \frac{\partial}{\partial z}\left(\mu \frac{\partial u_z}{\partial r}\right)$$

$$+ 2\frac{\partial \mu}{\partial z}\frac{\partial u_z}{\partial r} + \frac{\partial}{\partial z}\left(\mu \frac{\partial u_r}{\partial z}\right) = \rho \frac{\partial^2 u_r}{\partial t^2}$$

Vertical Component:

3.3 $$\frac{\partial}{\partial z}\left(\eta \frac{\partial u_z}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\eta}{r}\frac{\partial}{\partial r}(ru_r)\right) - \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial z}\right)$$

$$- 2\frac{\partial \mu}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) = \rho \frac{\partial^2 u_z}{\partial t^2}$$

It should be noted that in the original wave equation a body force term must also be included. For the case of a vertical point force at $(r,z)=(0,d)$ having time dependence $f(t)$

3.4
$$\vec{F}(r,z,t) = \frac{f(t)\delta(r)\delta(z-d)}{2\pi r}\hat{z}$$

and for an explosive source

3.5
$$\vec{F}(r,z,t) = f(t)\nabla\left\{\frac{\delta(r)\delta(z-d)}{2\pi r}\right\}.$$

Equations 3.2 and 3.3 determine the displacement components subject to initial conditions

3.6
$$\vec{u}\Big|_{t=0} = \frac{\partial\vec{u}}{\partial t}\Big|_{t=0} = 0$$

and boundary conditions at the free surface

3.7
$$\sigma_{zz}\Big|_{z=0} = 0$$

3.8
$$\tau_{rz}\Big|_{z=0} = 0$$

In cylindrical coordinates the boundary conditions at the free surface can be written as

3.9

$$\sigma_{zz} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + (\lambda+2\mu)\,\frac{\partial u_z}{\partial z}$$

3.10

$$\tau_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

As for the SH case, the dimensionality of the problem can be reduced through the introduction of finite Hankel transforms. The choice of the order of Hankel transform to be applied is determined largely by the form of the pair of differential equations which describe the motion, and the boundary conditions imposed at r=a.

The transforms appropriate to the boundary value problem expressed by 3.2 and 3.3 with boundary conditions $u_r|_{r=a} = 0$ and $\partial u_z/\partial r|_{r=a} = 0$ involves a consideration of the Sturm-Liouville transforms appropriate to this differential system. A finite first order Hankel transform is applied for u_r, and a finite zero order transform of the second kind is applied to u_z.

The transforms are given by

3.11
$$S(k_i,z,t) = \int_0^a u_r(r,z,t)J_1(k_ir)\,rdr$$

3.12
$$R(k_i,z,t) = \int_0^a u_z(r,z,t)J_o(k_ir)\,rdr$$

where k_i are the roots of $J_1(k_i a)=0$ (Sneddon (1972)).

Direct application of this pair of transforms to equations 3.2 and 3.3 reduces all operations with the variable "r" to scalar multiplication in the transform domain. Thus in P-SV problems with a point source on r=0, a zero order Hankel transform is appropriate for u_z and σ_{zz} whereas a first order transform is required for u_r and τ_{rz}. (Aki and Richards Vol I (1980)).

The inverse transform for the radial component is

3.13
$$u_r(r,z,t) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{S(k_i,z,t)J_1(k_i r)}{(J_2(k_i a))^2}$$

and for the vertical component

3.14
$$u_z(r,z,t) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{R(k_i,z,t)J_o(k_i r)}{(J_o(k_i a))^2}$$

The inversion for the radial component can be expressed in a form similar to that for the vertical component

3.15
$$u_r(r,z,t) = \frac{2}{a^2} \sum_{i=2}^{\infty} \frac{S(k_i,z,t)J_1(k_i r)}{(J_o(k_i a))^2}$$

where the relation

$$-J_2(k_i a) = J_o(k_i a) - \frac{2}{k_i a} (J_1(k_i a)$$

has been used (Abramowitz and Stegun (1968)).

As for the SH case the introduction of boundary conditions at r=a introduces a fictitious reflecting surface. The boundary r=a must be chosen sufficiently far that reflections from it do not influence the seismic response in the region of interest.

The transformation of each term in the equation for the radial component involves integration by parts:

$$\int_0^a \frac{\partial}{\partial r}\left(\eta \frac{\partial u_z}{\partial z}\right) J_1(k_i r) r dr = -\eta k_i \frac{\partial}{\partial z} R(k_i,z,t)$$

$$\int_0^a \frac{\partial}{\partial z}\left(\mu \frac{\partial u_z}{\partial r}\right) J_1(k_i r) r dr = -k_i \frac{\partial}{\partial z} (\mu R(k_i,z,t))$$

$$\int_0^a \frac{\partial \mu}{\partial z} \frac{\partial u_z}{\partial r} J_1(k_i r) r dr = -k_i \frac{\partial \mu}{\partial z} R(k_i,z,t)$$

$$\int_0^a \eta \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial}{\partial r} (r u_r)\right) J_1(k_i r) r dr = -\eta k_i^2 S(k_i,z,t).$$

The equation for the radial component becomes

3.16

$$\frac{\partial}{\partial z}\left(\mu \frac{\partial S}{\partial z}\right) - \eta k_i \frac{\partial R}{\partial z} + k_i \frac{\partial}{\partial z} (\mu R) - 2k_i \frac{\partial \mu}{\partial z} R - \eta k_i^2 S = \rho \frac{\partial^2 S}{\partial t^2}.$$

To transform the equation for the vertical component, consider the terms of equation 3.3.

$$\int_0^a \frac{\partial}{\partial z}\left(\frac{\eta}{r} \frac{\partial}{\partial r} (r u_r)\right) J_o(k_i r) r dr = k_i \frac{\partial}{\partial z} (\eta S(k_i,z,t)$$

$$\int_0^a \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u_r}{\partial z}\right) J_o(k_i r)\,r\,dr = \mu k_i \frac{\partial}{\partial z} S(k_i,z,t)$$

$$\int_0^a \frac{\partial \mu}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}(ru_r) J_o(k_i r)\,r\,dr = \frac{\partial \mu}{\partial z}\; k_i S(k_i,z,t)$$

$$\int_0^a \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u_z}{\partial r}\right) J_o(k_i r)\,r\,dr = -\mu k_i^2\, R(k_i,z,t)$$

The equation for the vertical component becomes upon transformation:

3.17

$$\frac{\partial}{\partial z}\left(\eta\,\frac{\partial R}{\partial z}\right) + k_i\frac{\partial}{\partial z}(\eta S) - \mu k_i\frac{\partial S}{\partial z} - 2k_i\frac{\partial \mu}{\partial z} S - \mu k_i^2\, R = \rho\,\frac{\partial^2 R}{\partial t^2}\,.$$

To determine the radial and vertical displacement components it is necessary to solve the coupled system given by equations 3.16 and 3.17 using finite-difference methods. The inverse transforms given by 3.14 and 3.15 define the components u_r and u_z respectively.

3.2 Finite-Difference Equations

It remains to develop finite-difference analogues for equations 3.16 and 3.17. In the equation for the radial component the terms in $R(k_i,z,t)$ can be written in the form (equation 3.16):

3.18 $$-k_i\left\{\eta\,\frac{\partial R}{\partial z} - \frac{\partial}{\partial z}(\mu R) + 2\,\frac{\partial \mu}{\partial z}\,R\right\}$$

$$= -k_i\left\{(\eta-\mu)\,\frac{\partial R}{\partial z} + \frac{\partial \mu}{\partial z}\,R\right\}$$

$$= - \frac{k_i}{4\Delta z} \Big\{ (\eta_{k+1}+\eta_{k-1}-\mu_{k+1}-\mu_{k-1})(R_{k+1}-R_{k-1})$$

$$+ (\mu_{k+1}-\mu_{k-1})(R_{k+1}-R_{k-1}) \Big\}$$

$$= - \frac{k_i}{2\Delta z} \left\{ \left[(\eta_k - 2\mu_k) + \mu_{k+1} \right] R_{k+1} - \left[(\eta_k - 2\mu_k) + \mu_{k-1}) \right] R_{k-1} \right\}.$$

Arithmetic averages have been used in order to provide some smoothing of the medium parameters and response. In a similar manner, the terms in equation 3.17 for the vertical component which involve $S(k_i, z, t)$ can be written:

3.19
$$k_i \left\{ \frac{\partial}{\partial z} (\eta S) - \mu \frac{\partial S}{\partial z} - 2 \frac{\partial \mu}{\partial z} S \right\}$$

$$= k_i \left\{ (\eta - \mu) \frac{\partial S}{\partial z} + \frac{\partial}{\partial z} (\eta - 2\mu) S \right\}$$

$$= \frac{k_i}{4\Delta z} \Big\{ (\eta_{k+1}+\eta_{k-1}-\mu_{k+1}-\mu_{k-1})(S_{k+1}-S_{k-1})$$

$$+ (\eta_{k+1}-2\mu_{k+1}-\eta_{k-1}+2\mu_{k+1})(S_{k+1}+S_{k-1}) \Big\}$$

$$= \frac{k_i}{2\Delta z} \left\{ \left[(\eta_{k+1} - 2\mu_{k+1}) + \mu_k \right] S_{k+1} - \left[(\eta_{k-1} - 2\mu_{k-1}) + \mu k \right] S_{k-1} \right\}$$

In the same manner as for the SH case, the following analogues are used:

3.20
$$\frac{\partial}{\partial z} \left(\mu(z) \frac{\partial S}{\partial z} \right) = \frac{a_k S_{k-1} - (a_k + a_{k+1}) S_k + a_{k+1} S_{k+1}}{(\Delta z)^2}$$

$$\text{where } a_k = \frac{2\mu_k \mu_{k-1}}{\mu_k + \mu_{k-1}}$$

3.21

$$\frac{\partial}{\partial z}\left(\eta(z)\ \frac{\partial R}{\partial z}\right) = \frac{b_k R_{k-1} - (b_k + b_{k+1}) R_k + b_{k+1} R_{k+1}}{(\Delta z)^2}$$

$$\text{where } b_k = \frac{2\eta_k \eta_{k-1}}{\eta_k + \eta_{k-1}}$$

3.22

$$\rho\ \frac{\partial^2 S}{\partial t^2} = \rho_k \left(\frac{S_k^{i+1} - 2S_k^i + S_k^{i-1}}{(\Delta t)^2}\right)$$

3.23

$$\gamma\ \frac{\partial S}{\partial t} = \gamma_k \left(\frac{S_k^{i+1} - S_k^{i-1}}{2(\Delta t)}\right)$$

The analogues for the time derivative in R are indentical to 3.22 and 3.23. Making use of these relations the finite-difference equivalent of equation 3.16 for the radial component is:

3.24

$$\frac{a_k S_{k-1}^i - (a_k + a_{k+1}) S_k^i + a_{k+1} S_{k+1}^i}{(\Delta z)^2} - \rho_k \left(\frac{S_k^{i+1} - 2S_k^i + S_k^{i-1}}{(\Delta t)^2}\right)$$

$$-k_i^2 \eta_k S_k^i - \frac{k_i}{2\Delta z}\left((\eta_k - 2\mu_k + \mu_{k+1}) R_{k+1}^i - (\eta_k - 2\mu_k + \mu_{k-1}) R_{k-1}^i\right)$$

$$-\ \gamma_k \left(\frac{S_k^{i+1} - S_k^{i-1}}{2\Delta t}\right) = 0.$$

This equation is to be solved for S_k^{i+1} explicitly and can be written as

3.25

$$\left(\frac{\rho_k}{(\Delta t)^2}+\frac{\gamma_k}{2\Delta t}\right)S_k^{i+1}=\frac{a_k S_{k+1}^i}{(\Delta z)^2}-\left(\frac{a_k+a_{k+1}}{(\Delta z)^2}+k_i^2\eta_k-\frac{2\rho_k}{(\Delta t)^2}\right)S_k^i$$

$$+\frac{a_k S_{k-1}^i}{(\Delta z)^2}-\frac{k_i}{2\Delta z}(\eta_k-2\mu_k+\mu_{k+1})R_{k+1}^i$$

$$+\frac{k_i}{2\Delta z}(\eta_k-2\mu_k+\mu_{k-1})R_{k-1}^i$$

$$-\left(\frac{\rho_k}{(\Delta t)^2}-\frac{\gamma_k}{2\Delta t}\right)S_k^{i-1}. \qquad 1\le k\le m$$

The finite-difference analogue of the equation for the vertical component is

3.26

$$\frac{b_k R_{k-1}^i-(b_k+b_{k+1})R_k^i+b_{k+1}R_{k+1}^i}{(\Delta z)^2}-\rho_k\left(\frac{R_k^{i+1}-2R_k^i+R_k^{i-1}}{(\Delta t)^2}\right)$$

$$-k_i^2\mu_k R_k^i+\frac{k_i}{2\Delta z}(\eta_{k+1}-2\mu_{k+1}+\mu_k)S_{k+1}^i-(\eta_{k-1}-2\mu_{k-1}+\mu_k)S_{k-1}^i$$

$$-\gamma_k\left(\frac{R_k^{i+1}-R_k^{i-1}}{2\Delta t}\right)=0$$

or equivalently as

3.27

$$\left(\frac{\rho_k}{(\Delta t)^2} + \frac{\gamma_k}{2\Delta t}\right) R_k^{i+1} = \frac{b_{k+1}R_{k+1}^i}{(\Delta z)^2} - \left(\frac{b_k+b_{k+1}}{(\Delta z)^2} + k_i^2\mu_k - \frac{2\rho_k}{(\Delta t)^2}\right) R_k^i$$

$$+ \frac{b_k R_{k-1}^i}{(\Delta z)^2} + \frac{k_i}{2\Delta z}\left(\eta_{k+1} - 2\mu_{k+1} + \mu_k\right) S_{k+1}^i$$

$$- \frac{k_i}{2\Delta z}\left(\eta_{k-1} - 2\mu_{k-1} + \mu_k\right) S_{k-1}^i$$

$$- \left(\frac{\rho_k}{(\Delta t)^2} - \frac{\gamma_k}{2\Delta t}\right) R_k^{i-1}. \qquad 1 \le k \le m$$

Equations 3.25 and 3.27 are used to evaluate successive values of $R(k_i,z,t)$ and $S(k_i,z,t)$. The displacements u_r and u_z are found from the inverse transforms given by equations 3.14 and 3.15. It should be noted that the body force terms must also be incorporated into the finite-difference equations.

3.3 Source Terms

For the case of a vertical point force, the transform of the body force term is included on the left side of equation 3.17 for the vertical component. The transform is given by

3.28

$$\int_0^a F(r,z,t)J_o(k_i r)rdr$$

$$= \frac{f(t)\delta(z-d)}{2\pi}\int_0^a \delta(r)J_o(k_i r)dr$$

$$= \frac{f(t)\delta(z-d)}{4\pi}\ .$$

This term is to be added to $R(k_i,z,t)$ at depth $z=d$. The Dirac delta function in this case has been approximated by $1/\Delta z$ since $\delta(z-d)\cdot\Delta z=1$. The function $f(t)$ is an exponentially damped sine function as for the SH case.

For an explosive source

3.29

$$\vec{F}(r,z,t) = f(t)\ \nabla\left\{\frac{\delta(r)\delta(z-d)}{2\pi r}\right\}$$

$$= f(t)\ \frac{\partial}{\partial r}\left(\frac{\delta(r)\delta(z-d)}{2\pi r}\right)\hat{r} + f(t)\ \frac{\partial}{\partial z}\left(\frac{\delta(r)\delta(z-d)}{2\pi r}\right)\hat{z}.$$

The transform of the radial component is

3.30

$$\int_0^a f(t)\ \frac{\partial}{\partial r}\left(\frac{\delta(r)\delta(z-d)}{2\pi r}\right)J_1(k_i r)rdr$$

$$= -\ \frac{k_i f(t)\delta(z-d)}{4\pi\Delta z}\ .$$

This term is to be added to $S(k_i,z,t)$ at depth $z=d$. The

transform of the vertical component for this case is

3.31

$$\int_0^a f(t) \frac{\partial}{\partial z}\left(\frac{\delta(r)\delta(z-d)}{2\pi r}\right) J_o(k_i r) r dr$$

$$= \frac{\partial}{\partial z}\, \delta(z-d)\, \frac{f(t)}{4\pi} \,.$$

If $\delta(z-d)$ is approximated by $1/\Delta z$ at depth $z=k\Delta z$ then an approximation to $\partial/\partial z\ \delta(z-d)$ is given by $1/(\Delta z)^2$ at $z=(k-1)\Delta z$ and $-1/(\Delta z)^2$ at $z=(k+1)\Delta z$. The terms to be added to $R(k_i,z,t)$ are

3.32

$$\frac{f(t)}{4\pi(\Delta z)^2} \quad \text{at} \quad z = (k-1)\Delta z$$

$$-\frac{f(t)}{4\pi(\Delta z)^2} \quad \text{at} \quad z = (k+1)\Delta z$$

for the duration of the seismic pulse.

3.4 Free Surface Condition

The finite-difference equations given by 3.25 and 3.27, with the incorporation of the appropriate body force terms, are to be solved for all grid steps k where $1 \le k \le m$. At the free surface, $k=0$, the assumption of a homogeneous medium allows the equations of motion to be written in a simplified form. Homogeneity for $k=0,1,2$ enables equations 3.16 and 3.17 to be written as

3.33
$$v_s^2 \frac{\partial^2 S}{\partial z^2} - k_i(v_p^2 - v_s^2)\frac{\partial R}{\partial z} - k_i^2 v_p^2 S = \frac{\partial^2 S}{\partial t^2}$$

3.34
$$v_p^2 \frac{\partial^2 R}{\partial z^2} + k_i(v_p^2 - v_s^2)\frac{\partial S}{\partial z} - k_i^2 v_s^2 R = \frac{\partial^2 R}{\partial t^2}$$

where $v_s^2 = \mu/\rho$ and $v_p^2 = (\lambda+2\mu)/\rho$. If finite-difference analogues are written for these equations, they will contain the fictitious values R_{-1}^{j} and S_{-1}^{j}. In order to eliminate these values from the equations, the boundary conditions at the free surface given by equations 3.7 and 3.10 may be used. From equations 3.7 and 3.9 we may write, for σ_{zz}

$$\lambda\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \eta\frac{\partial u_z}{\partial z} = 0.$$

Application of a zero order finite Hankel transform yields for z=0:

$$\int_0^a \lambda\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)J_0(k_i r)\,r\,dr + \int_0^a \eta\frac{\partial u_z}{\partial z}J_0(k_i r)\,r\,dr = 0$$

$$\lambda k_i S(k_i,z,t) + \eta\frac{\partial R}{\partial z}(k_i,z,t) = 0$$

3.35 $$v_p^2 \frac{\partial R}{\partial z} = -k_i(v_p^2 - 2v_s^2)S.$$

The finite-difference analogue to this last equation yields

3.36

$$R_{-1}^{i} = R_{1}^{i} + 2k_i \Delta z \left(1 - \frac{2v_s^2}{v_p^2}\right) S_o^i .$$

In a similar manner, the equation for the shear stress at the free surface given by 3.8 and 3.10 can be used to determine a substitution for the fictitious value S_{-1}^{i}. Application of the transform yields

$$\mu \left\{ \int_o^a \frac{\partial u_z}{\partial r} J_1(k_i r) r dr + \int_o^a \frac{\partial u_r}{\partial z} J_1(k_i r) r dr \right\} = 0$$

3.37 $$-k_i R(k_i,z,t) + \frac{\partial S}{\partial z}(k_i,z,t) = 0.$$

The finite-difference analogue to this free surface stress condition is

3.38

$$S_{-1}^{i} = S_{1}^{i} - 2k_i \Delta z R_o^i .$$

From 3.33 the finite-difference relation for z=0 may be obtained as

$$v_s^2 \left(\frac{\partial^2 S}{\partial z^2}\right)_o^i - k_i (v_p^2 - v_s^2\left(\frac{\partial R}{\partial z}\right)_o^i - k_i^2 v_p^2 S_o^i = \left(\frac{\partial^2 S}{\partial t^2}\right)_o^i$$

$$v_s^2 \frac{(S_{-1}^i - 2S_o^i + S_1^i)}{(\Delta z)^2} + k_i^2 \left\{ (v_p^2 - v_s^2) \left(1 - \frac{2v_s^2}{v_p^2} \right) - v_p^2 \right\} S_o^i = \frac{S_o^{i+1} - 2S_o^i + S_o^{i-1}}{(\Delta t)^2} .$$

The explicit solution for S_o^{i+1} becomes

$$S_o^{i+1} = S_1^i \left(\frac{v_s^2 (\Delta t)^2}{(\Delta z)^2} \right) + S_o^i \left\{ 2 - \frac{2v_s^2 (\Delta t)^2}{(\Delta z)^2} + (\Delta t)^2 k_i^2 \left((v_p^2 - v_s^2) \right. \right.$$

$$\left. \left. \left(1 - \frac{2v_s^2}{v_p^2} \right) - v_p^2 \right) \right\} + S_{-1}^i \left(\frac{v_s^2 (\Delta t)^2}{(\Delta z)^2} \right) - S_o^{i-1} .$$

Substitution of equation 3.38 to eliminate S_{-1}^i, we have 3.39

$$S_o^{i+1} = S_1^i \left(\frac{2v_s^2 (\Delta t)^2}{(\Delta z)^2} \right) + S_o^i \left\{ 2 - \frac{2v_s^2 (\Delta t)^2}{(\Delta z)^2} + (\Delta t)^2 k_i^2 \left((v_p^2 - v_s^2) \right. \right.$$

$$\left. \left. \left(1 - \frac{2v_s^2}{v_p^2} \right) - v_p^2 \right) \right\} + R_o^i \left(- \frac{2k_i (\Delta t)^2 v_s^2}{\Delta z} \right) - S_o^{i-1} .$$

The explicit solution for R_o^{i+1} is obtained from equation 3.34 in the same way:

$$\left(v_p^2 \frac{\partial^2 R}{\partial z^2} \right)_o^i + k_i (v_p^2 - v_s^2) \left(\frac{\partial S}{\partial z} \right)_o^i - k_i^2 v_s^2 R_o^i = \left(\frac{\partial^2 R}{\partial t^2} \right)_o^i$$

$$v_p^2 \frac{(R_{-1}^i - 2R_o^i + R_1^i)}{(\Delta z)^2} + k_i^2 (v_p^2 - 2v_s^2) R_o^i = \frac{R_o^{i+1} - 2R_o^i + R_o^{i-1}}{(\Delta t)^2} .$$

The explicit solution for R_o^{i+1} is

$$R_o^{i+1} = 2R_o^i - R_o^{i-1} + (\Delta t)^2 \left\{ \frac{v_p^2 (R_{-1}^i - 2R_o^i + R_1^i)}{(\Delta z)^2} + k_i^2 (v_p^2 - 2v_s^2) R_o^i \right\} .$$

Substitution of equation 3.36 for R_{-1}^i yields

3.40

$$R_o^{i+1} = S_o^i \left\{ 2\Delta z k_i \left(1 - \frac{2v_s^2}{v_p^2}\right) \frac{(\Delta t)^2 v_p^2}{(\Delta z)^2} \right\} + R_1^i \left(\frac{2v_p^2 (\Delta t)^2}{(\Delta z)^2} \right)$$

$$+ R_o^i \left\{ 2 - \frac{2v_p^2 (\Delta t)^2}{(\Delta z)^2} + (\Delta t)^2 k_i^2 \, (v_p^2 - 2v_s^2) \right\} - R_o^{i-1} .$$

Making use of the relations given by 3.39 and 3.40, the truncation error at the free surface as well as throughout the interior of the grid is of the order $O((\Delta z)^2, (\Delta t)^2)$. The truncation error at the free surface was thought to be responsible for the generation of the S* non-geometric arrival identified by Hron and Mikhailenko (1981). A Taylor series expansion at the free surface for $R(k_i, z, t)$ and $S(k_i, z, t)$ was then used to reduce the truncation error to the order of approximation $O((\Delta z)^4, (\Delta t)^2)$. The incorporation of this improved boundary condition did not alter the results substantially and the nature of the arrivals was unchanged. The basic features of the S* arrival were first inferred by Hron and Mikhailenko (1981) from synthetic seismograms based on the Alekseev-Mikhailenko method.

3.5 Stability

In order to determine the stability condition for the P-SV case the coupled finite-difference equations to be analyzed by the von Neumann method are given by equations 3.25 and 3.27.

In the same manner as for the SH case, if we assume that $\gamma_k=0$ and set the variable coefficients to their values at a specific grid point, the modified equations in terms of shear and compressional velocities are

3.41

$$S_k^{i+1} = \left(\frac{v_s \Delta t}{\Delta z}\right)^2 S_{k+1}^i - \left\{2\left(\frac{v_s \Delta t}{\Delta z}\right)^2 - 2 + k_i^2 v_p^2 (\Delta t)^2\right\} S_k^i + \left(\frac{v_s \Delta t}{\Delta z}\right)^2 S_{k-1}^i - k_i (\Delta t)^2 (v_p^2 - v_s^2) \frac{(R_{k+1}^i - R_{k-1}^i)}{2\Delta z} - S_k^{i-1}$$

3.42

$$R_k^{i+1} = \left(\frac{v_p \Delta t}{\Delta z}\right)^2 R_{k+1}^i - \left\{2\left(\frac{v_p \Delta t}{\Delta z}\right)^2 - 2 + k_i^2 v_s^2 (\Delta t)^2\right\} R_k^i + \left(\frac{v_p \Delta t}{\Delta z}\right)^2 R_{k-1}^i + k_i (\Delta t)^2 (v_p^2 - v_s^2) \frac{(S_{k+1}^i - S_{k-1}^i)}{2\Delta z} - R_k^{i-1}$$

where $v_s^2 = \mu/\rho$ and $v_p^2 = \eta/\rho$.

For a system of coupled equations, Richtmyer and Morton (1967) dicuss in detail the basis for using a harmonic decomposition in terms of a Fourier series. In our particular case it will be necessary to determine the amplification matrix from which the characteristic equation

yields the eigenvalues. To ensure stability the spectral radius, or maximum of $|\lambda_i|$ i=1,2,3,4 must be uniformly bounded: $|\lambda_i|<1$.

Given a properly posed initial value problem and a finite-difference approximation to it, Lax's Equivalence Theorem states that stability is the necessary and sufficient condition for convergence. The consistency condition requires that the truncation error $\to 0$ as $(\Delta t)\to 0$ for $0\le t\le T$. The amplification factors or eigenvalues can be obtained by substituting the error function directly into the difference equations. This yields a system of linear equations for R_k^{i+1} and S_k^{i+1} and the vanishing of their determinant determines $\vec{\lambda}$. In our case, the coupled equations given by 3.41 and 3.42 can be expressed as

3.43

$$S_k^{i+1} = \left(\frac{v_s \Delta t}{\Delta z}\right)^2 S_{k+1}^i - \left\{2\left(\frac{v_s \Delta t}{\Delta z}\right)^2 - 2 + k_i^2 v_p^2 (\Delta t)^2\right\} S_k^i$$
$$+ \left(\frac{v_s \Delta t}{\Delta z}\right)^2 S_{k-1}^i - k_i (\Delta t)^2 (v_p^2 - v_s^2) \frac{(R_{k+1}^i - R_{k-1}^i)}{2\Delta z} - P_k^i$$

3.44

$$R_k^{i+1} = \left(\frac{v_p \Delta t}{\Delta z}\right)^2 R_{k+1}^i - \left\{2\left(\frac{v_p \Delta t}{\Delta z}\right)^2 - 2 + k_i^2 v_s^2 (\Delta t)^2\right\} R_k^i$$
$$+ \left(\frac{v_p \Delta t}{\Delta z}\right)^2 R_{k-1}^i + k_i (\Delta t)^2 (v_p^2 - v_s^2) \frac{(S_{k+1}^i - S_{k-1}^i)}{2\Delta z} - Q_k^i$$

where $P_k^{i+1} = S_k^i$ and $Q_k^{i+1} = R_k^i$ have been used to reduce the

system to a two-level scheme. If we write

$$S_{k+1}^{i} = e^{i\beta\Delta z}\, S_{k}^{i}$$

$$S_{k-1}^{i} = e^{-i\beta\Delta z}\, S_{k}^{i}$$

$$R_{k+1}^{i} = e^{i\beta\Delta z}\, R_{k}^{i}$$

$$R_{k-1}^{i} = e^{-i\beta\Delta z}\, R_{k}^{i}$$

then the set of equations can be written in the form:

3.45

$$P_{k}^{i-1} = S_{k}^{i}$$

$$Q_{k}^{i+1} = R_{k}^{i}$$

$$R_{k}^{i+1} = -Q_{k}^{i} + \left\{\left(\frac{v_p\Delta t}{\Delta z}\right)^2 (2\cos(\beta\Delta z) - 2) + 2 - k_i^2 v_s^2 (\Delta t)^2\right\} R_{k}^{i}$$

$$+ \frac{k_i(\Delta t)^2 (v_p^2 - v_s^2)}{\Delta z} (i\sin(\beta\Delta z)) S_{k}^{i}$$

$$S_{k}^{i+1} = -P_{k}^{i} - \frac{k_i(\Delta t)^2 (v_p^2 - v_s^2)}{\Delta z} (i\sin(\beta\Delta z)) R_{k}^{i}$$

$$+ \left\{\left(\frac{v_s\Delta t)}{\Delta z}\right)^2 (2\cos(\beta\Delta z) - 2) + 2 - k_i^2 v_p^2 (\Delta t)^2\right\} S_{k}^{i}.$$

In terms of an amplification matrix, $\vec{W}=(P,Q,R,S)$ can be expressed as:

3.46
$$\vec{W}_k^{i+1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & b \\ -1 & -1 & c & d \end{bmatrix} = \vec{W}_k^{i}$$

where

$$a = \left(\frac{v_p \Delta t}{\Delta z}\right)^2 \quad (-4\sin^2(\beta\Delta z/2)) + 2 - k_i^2 v_s^2 (\Delta t)^2$$

$$b = \frac{ik_i(\Delta t)^2}{\Delta z}(v_p^2 - v_s^2)\sin(\beta\Delta z)$$

3.47
$$c = -b$$

$$d = \left(\frac{v_s \Delta t}{\Delta z}\right)^2 \quad (-4\sin^2(\beta\Delta z/2)) + 2 - k_i^2 v_p^2 (\Delta t)^2 \; .$$

The eigenvalues of the amplification matrix in equation 3.46 are given by $|A-\lambda I|=0$. The amplification factors are found from the roots of

3.48
$$\lambda^4 - (a+d)\lambda^3 + (2+ad-bc)\lambda^2 - (a+d)\lambda + 1 = 0.$$

If we denote the eigenvalues as $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ then the equation of degree 4 having these roots can be written

3.49
$$\lambda^4 - (\lambda_1+\lambda_2+\lambda_3+\lambda_4)\lambda^3$$
$$+ (\lambda_1\lambda_2+\lambda_3\lambda_4 + (\lambda_1+\lambda_2)(\lambda_3+\lambda_4))\lambda^2$$
$$- (\lambda_1\lambda_2(\lambda_3+\lambda_4) + \lambda_3\lambda_4(\lambda_1+\lambda_2))\lambda$$
$$+ \lambda_1\lambda_2\lambda_3\lambda_4 = 0.$$

Comparison of 3.48 and 3.49 yields the relation

3.50

$$\sum_{i=1}^{4} \lambda_i = a + d$$

$$= 4 - 4\sin^2(\beta\Delta z/2)\left\{\left(\frac{v_p\Delta t}{\Delta z}\right)^2 + \left(\frac{v_s\Delta t}{\Delta z}\right)^2\right\} - k_i^2(\Delta t)^2(v_s^2+v_p^2).$$

The requirement for stability is $|\lambda_i|<1$. Imposing this condition on the eigenvalues we have

3.51

$$\left|\sum_{i=1}^{4} \lambda_i\right| \le \sum_{i=1}^{4} |\lambda_i|$$

$$< 4.$$

From equation 3.50 the requirement for stability is

$$\left|4-4\sin^2(\beta\Delta z/2)\left\{\left(\frac{v_p\Delta t}{\Delta z}\right)^2 + \left(\frac{v_s\Delta t}{\Delta z}\right)^2\right\} - k_i^2(\Delta t)^2(v_s^2+v_p^2)\right| < 4$$

$$\left|4-4\left\{\left(\frac{v_p\Delta t}{\Delta z}\right)^2 + \left(\frac{v_s\Delta t}{\Delta z}\right)^2\right\} - k_i^2(\Delta t)^2(v_s^2+v_p^2)\right| < 4$$

The two equivalent inequalities are

$$\left\{\left(\frac{v_p\Delta t}{\Delta z}\right)^2 + \left(\frac{v_s\Delta t}{\Delta z}\right)^2\right\} + \frac{k_i^2(\Delta t)^2}{4}(v_p^2 + v_s^2) > 0$$

$$\left\{\left(\frac{v_p \Delta t}{\Delta z}\right)^2 + \left(\frac{v_s \Delta t}{\Delta z}\right)^2\right\} + \frac{k_i^2 (\Delta t)^2}{4} (v_p^2 + v_s^2) < 2$$

The first inequality is trivially satisfied since all terms are positive definite. The necessary and sufficient condition for stabilty of the finite-difference scheme is given by the last inequality.

4. NUMERICAL MODELS

In order to provide a comparison of results with those presented in the literature, a particular model has been duplicated for the case of SH waves and then extended for other source types. The model used for the calculation of synthetic seismograms for a vertically inhomogeneous medium is one presented by M. Korn and G. Müller (1983) in which two thin low-velocity layers representing coal seams are embedded in a homogeneous half-space. In their paper SH waves are generated by a horizontal single force at the surface and the results confirmed by comparison with the ray reflectivity method.

The statement of the problem is given in cylindrical coordinates (r,ϕ,z) with the z-axis pointing downward into the vertically inhomogeneous half-space whose surface is at $z=0$. As before the material parameters are the density $\rho(z)$, rigidity $\mu(z)$ and S velocity $\beta(z)$. At the surface a horizontal single force $f(t)$ acts in the direction $\phi=0$. The boundary values of the tangential stresses are

4.1
$$\tau_{zr} = -\cos\phi \, \frac{\delta(r)}{2\pi r} \, f(t)$$

4.2
$$\tau_{z\phi} = \sin\phi \, \frac{\delta(r)}{2\pi r} \, f(t).$$

The differential equation describing u_ϕ, the azimuthal displacement is

4.3
$$\rho \frac{\partial^2 u_\phi}{\partial t^2} = \mu \left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial u_\phi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial u_\phi}{\partial z} \right).$$

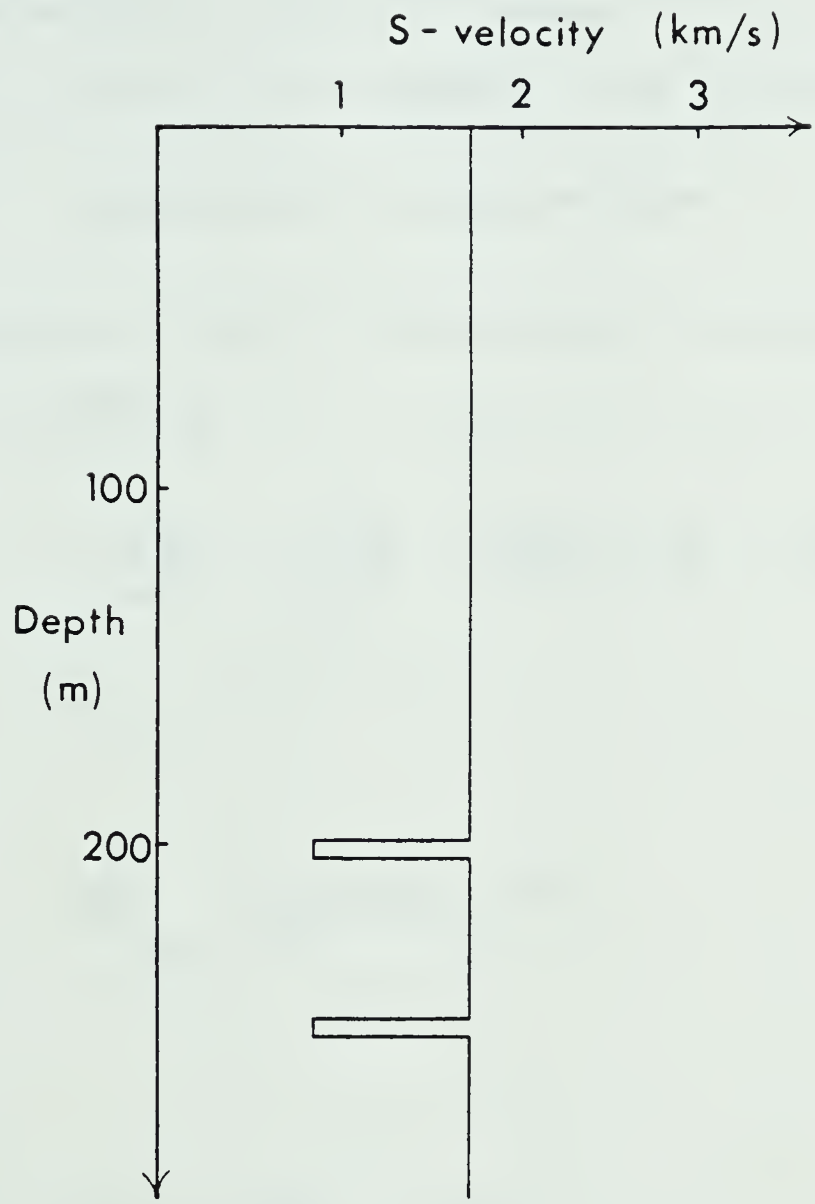


Figure 4.1 Velocity vs depth for the coal seam model.

The stress relation $\tau_{z\phi} = \mu(\partial u_\phi/\partial z)$ together with 4.2 yields

4.4
$$\left.\frac{\partial u_\phi}{\partial z}\right|_{z=0} = \frac{\sin\phi\ \delta(r)}{2\pi r\ \ \mu(o)} f(t).$$

Since the azimuth ϕ does not appear in 4.3, equation 4.4 implies $u_\phi \sim \sin\phi$ for constant r and z. Hence

4.5
$$u_\phi(r,\phi,z,t) = u(r,z,t)\sin\phi.$$

As before, the initial-boundary-value problem to be solved is given by

4.6
$$\rho \frac{\partial^2 u}{\partial t^2} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\ \ \mu\frac{\partial u}{\partial z}$$

4.7
$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{\delta(r)}{2\pi r\ \mu(o)} f(t)$$

4.8
$$\left.u\right|_{z=0} = \left.\frac{\partial u}{\partial t}\right|_{z=0} = 0.$$

The finite Hankel transform pair used in reducing the dimensionality of the problem is given by

4.9
$$S(k_i,z,t) = \int_o^a u(r,z,t)J_o(k_ir)rdr$$

4.10
$$u(r,z,t) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{S(k_i,z,t) J_o(k_i r)}{(J_1(k_i a))^2}$$

where k_i are the roots of $J_0(k_i a)=0$. The differential equation for the Hankel transform $S(k_i,z,t)$ and the boundary and initial conditions are given by

4.11
$$\rho \frac{\partial^2 S}{\partial t^2} = \frac{\partial}{\partial z}\left(\mu(z) \frac{\partial S}{\partial z}\right) - k_i^2 \mu(z) S$$

4.12
$$\left.\frac{\partial S}{\partial z}\right|_{z=0} = \frac{f(t)}{2\pi \mu(o)}$$

4.13
$$\left. S \right|_{z=0} = \left.\frac{\partial S}{\partial t}\right|_{z=0} = 0.$$

Upon solution of this transformed initial-boundary-value problem, the displacement is given by the inverse transform 4.10.

For the numerical calculations, the coal seam model involves layers each 2m thick having density 1.6 g/cm^3 and S-velocity 866 m/s embedded in a half-space of density 2.6 g/cm^3 and S-velocity 1732 m/s. The coal seams are placed at

depths of 200m and 250m. On a profile at the surface, along $\phi=90^{\circ}$, one expects direct waves, primary seam reflections, interseam multiples, and multiples between the surface and the seams.

In order to match the arrivals given in their paper, the dominant frequency of the force f(t) was taken to be 60 Hz rather than 30 Hz as stated by Korn and Müller. In the seismograms presented here, epicentral distances and depths are given in terms of S-wavelengths for the half-space. In all figures, times are given in periods rather than seconds. With these changes the coal seams are at depths of 6.90 WL and 8.625 WL. One set of 10 receivers is located on the surface of the half-space at epicentral increments of 0.862 WL. For the vertical seismic profiles another set of 24 receivers at equal depth increments of 0.375 WL has been placed at an epicentral distance of 4.31 WL. For the P-SV case the source is located at a depth of 0.75 WL. Sources for the SH case are invariably located on the surface.

4.1 Horizontal Point Force - Surface Traces

In Figure 4.3, the direct arrival A as well as the primary reflections B and C are evident. In order to identify each of the remaining arrivals, the direct arrival has been windowed out and the later arrivals amplified as shown in Figure 4.4. The primary reflections B and C, interseam multiples D and E, and the secondary reflections F and G have been identified and confirmed by matching travel

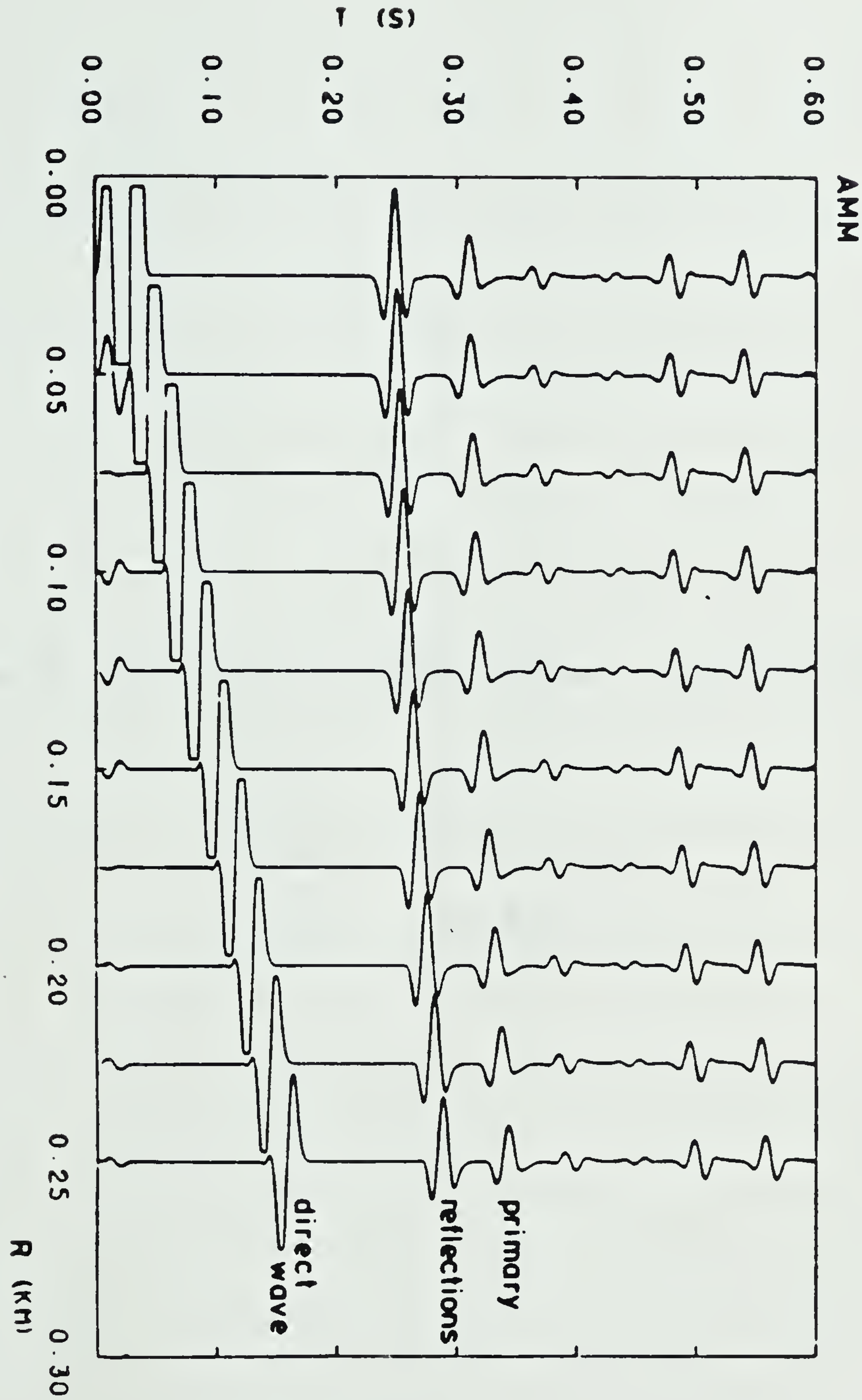
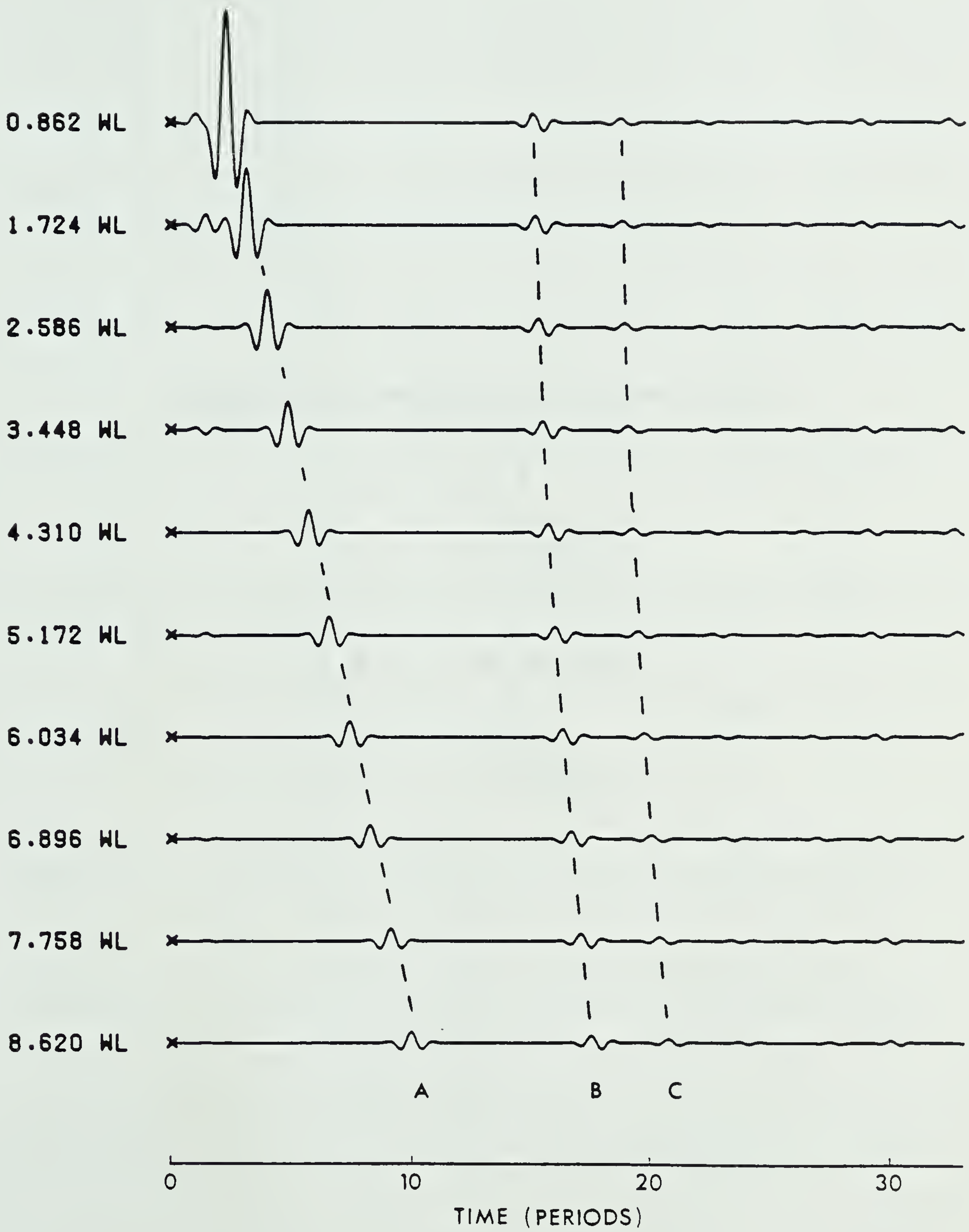


Figure 4.2 Traces from Korn and Müller.

Figure 4.3 Horizontal Point Force - Surface Traces

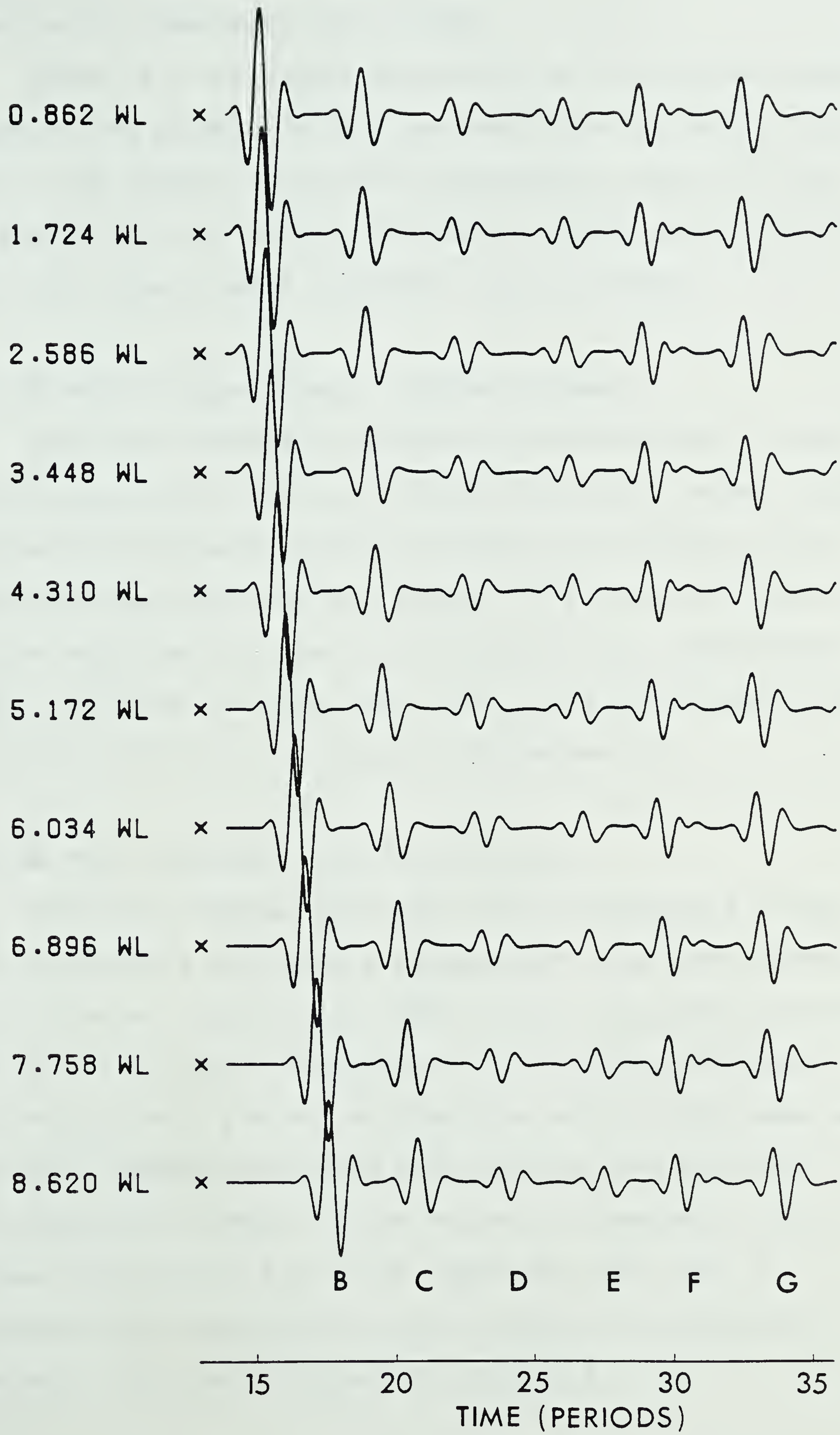
A	S_1
B	$S_1{}^2$
C	$S_1\ S_2{}^2\ S_1$



0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
A
B
C
0
10
20
30
TIME (PERIODS)

Figure 4.4 Horizontal Point Force

B	S_1^2
C	$S_1 S_2^2 S_1$
D	$S_1 S_2^4 S_1$
E	$S_1 S_2^6 S_1$
F	S_1^4
G	$S_1^3 S_2^2 S_1$


0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
B
C
D
E
F
G
15
20
25
30
35
TIME (PERIODS)

times using elementary ray methods.

There is a noticeable distortion of the original pulse shape in the primary reflections and later arrivals. This is due to the complex interference phenomena present in the wavelets which are reflected from or transmitted through the very thin layers which represent the coal seams.

4.2 SH Point Torque Source - Surface Traces

Only the windowed seismograms are presented in Figure 4.5 for this case. The significant difference between these arrivals and those for the horizontal point force is the amplitude dependence of the arrival on the angle θ where θ is the angle made by the ray with the z-axis. Ben-Menahem (1981) confirms that the amplitude of the disturbance varies directly with $\sin\theta$ for a point torque source.

4.3 SH Point Sources - Vertical Profiles

From the vertical profiles given in Figures 4.6 and 4.7, the nature of the wave propagation is evident. Energy may arrive at a particular depth either from above or below. The direct arrival A represents the downgoing wavefield; arrivals B and C are reflections from each of the seams and represent upward travelling disturbances. The interseam multiples are a result of the reflection labeled D. The primary reflection B from the uppermost coal seam is reflected once again at the free surface and propagates downward into the half-space as arrival E.

Figure 4.5 SH Torque Point Source

B	S_1^2
C	$S_1 S_2^2 S_1$
D	$S_1 S_2^4 S_1$
E	$S_1 S_2^6 S_1$
F	S_1^4
G	$S_1^3 S_2^2 S_1$

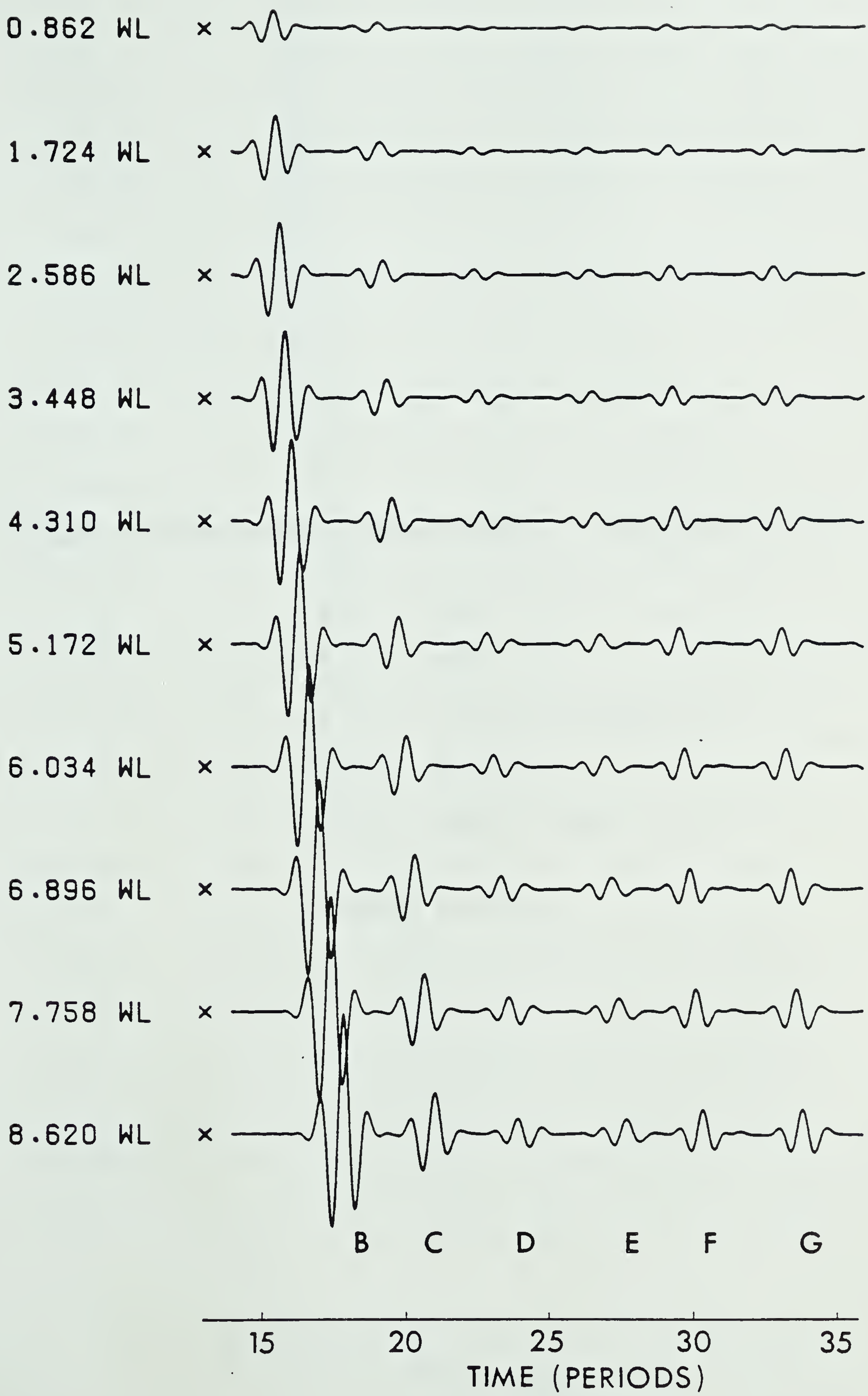

0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
B
C
D
E
F
G
15
20
25
30
35
TIME (PERIODS)

Figure 4.6 Horizontal Point Force - Vertical Profile

A	S_1
B	$S_1{}^2$
C	$S_1\ S_2{}^2$, $S_1\ S_2{}^2\ S_1$
D	$S_1\ S_2{}^3$
E	$S_1{}^3$, $S_1{}^3\ S_2$

Offset 4.31 WL

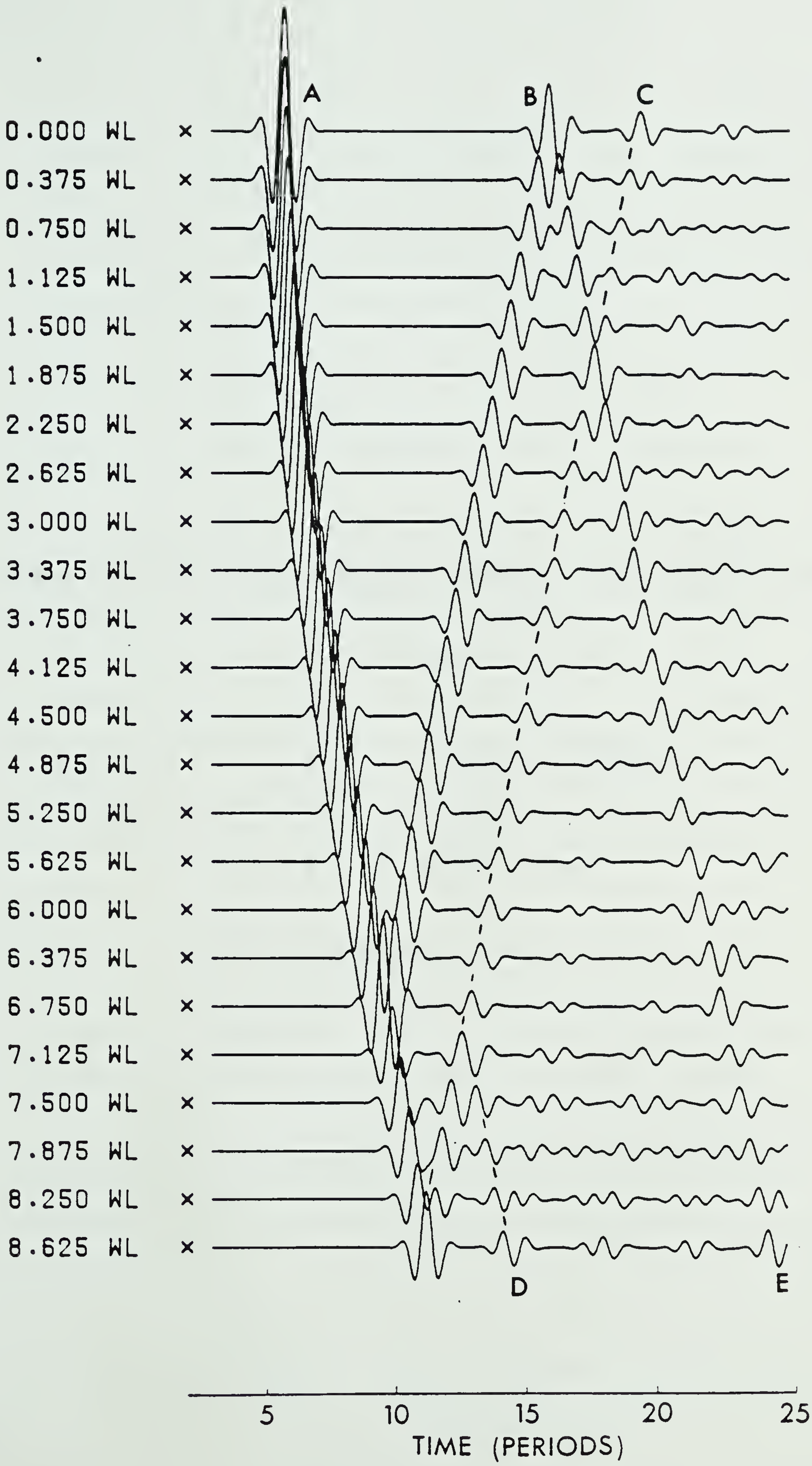
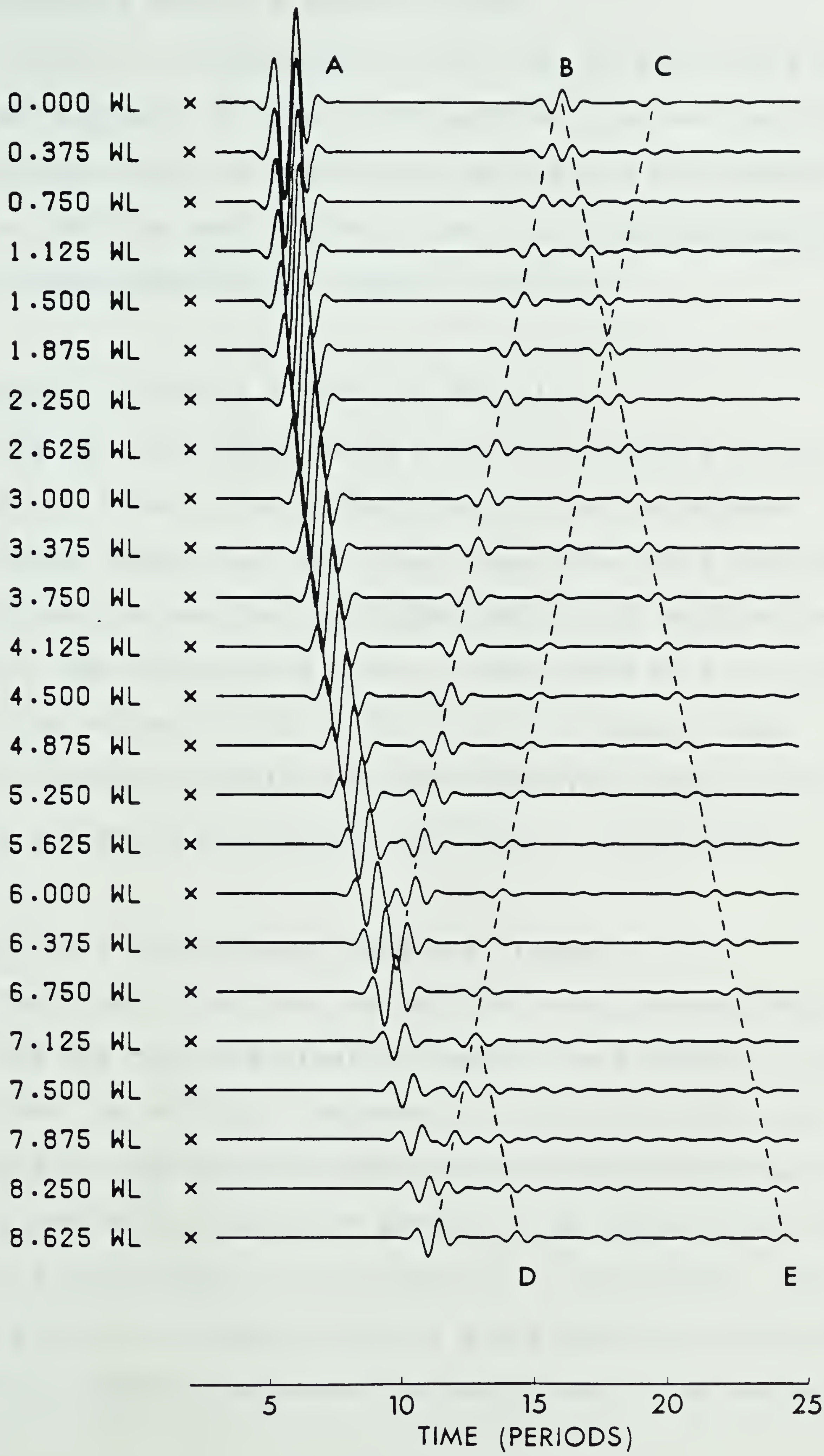

A
B
C
0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
D
E
5
10
15
20
25
TIME (PERIODS)

Figure 4.7 SH Torque Point Source - Vertical Profile

A	S_1
B	S_1^2
C	$S_1 S_2^2$, $S_1 S_2^2 S_1$
D	$S_1 S_2^3$
E	S_1^3, $S_1^3 S_2$

Offset 4.31 WL


A
B
C
0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
D
E
5
10
15
20
25
TIME (PERIODS)

4.4 Explosive Source - Surface Traces

The set of seismograms at the free surface are given in Figures 4.8 and 4.9. The direct arrival A as well as the reflections from the seams given by C and D are prominent in Figure 4.8. The weak arrival B is a low amplitude Rayleigh wave as confirmed by its apparent velocity.

4.5 Explosive Source - Vertical Profiles

The vertical profiles as given in Figures 4.10 and 4.11 illustrate clearly the propagation of the disturbance in the half-space. The direct P arrival identified as A undergoes significant conversion to SV upon reflection at the free surface. The resulting S wave is identified by B. Arrivals C and D are reflections from the first coal seam, their arrival times at the free surface matching those given in Figure 4.8 which were again confirmed by ray methods.

4.6 Vertical Point Force - Surface Traces

The direct P arrival as well as a very strong Rayleigh wave are the most discernable features in Figures 4.12 and 4.13. For the vertical component in particular, as given in Figure 4.13, the Rayleigh wave is the outstanding feature. The nature of the radiation pattern from the vertical point force is responsible in this case for its strength. The force acts as a source of both P and S waves and the strong vertical component enhances the magnitude of the surface wave.

Figure 4.8 Explosive Source - Horizontal Component of Surface Traces

A	P_1
B	Rayleigh Wave
C	$\grave{P}_1\ S_1$
D	$\acute{P}_1\ S_1{}^2$

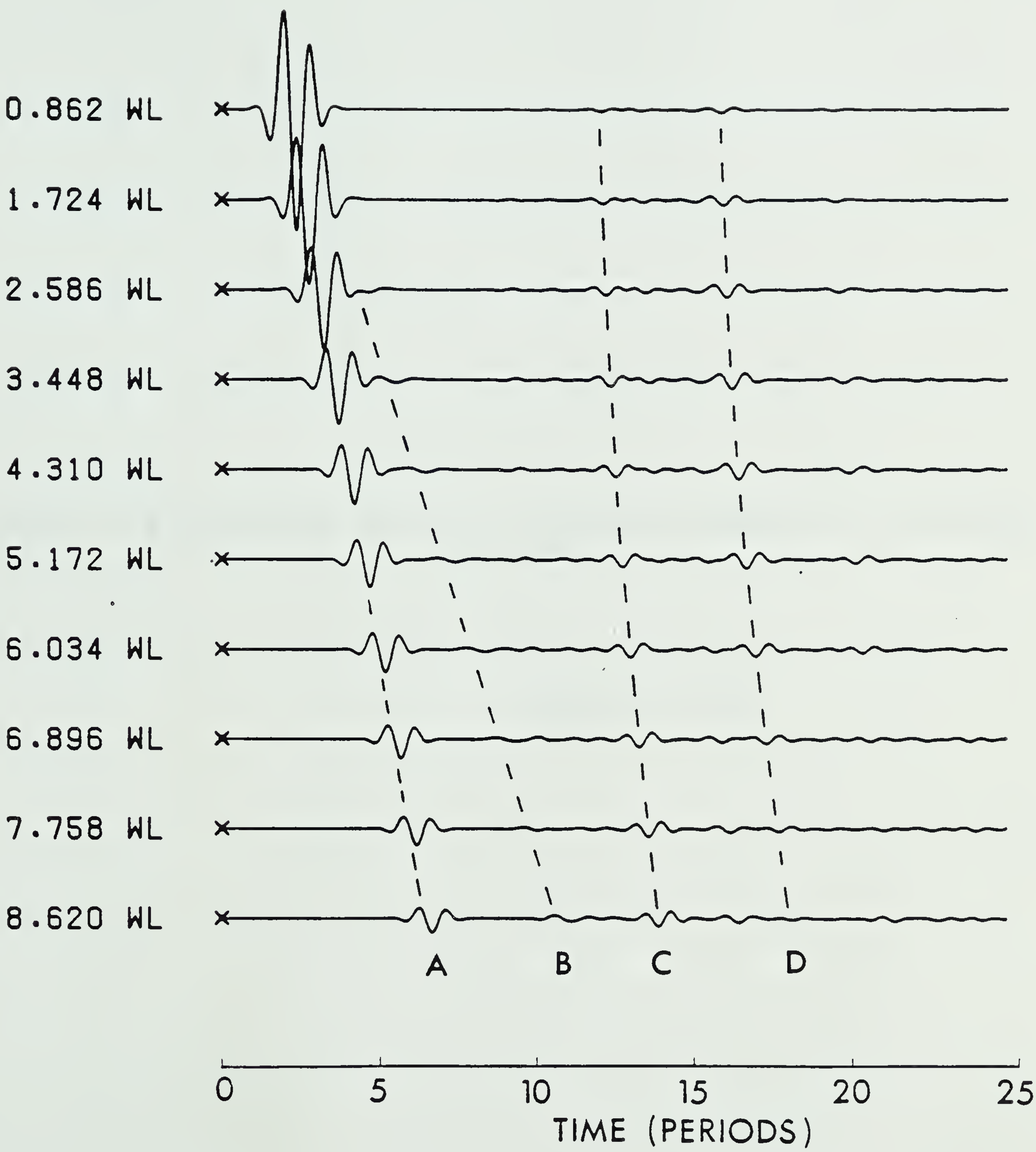
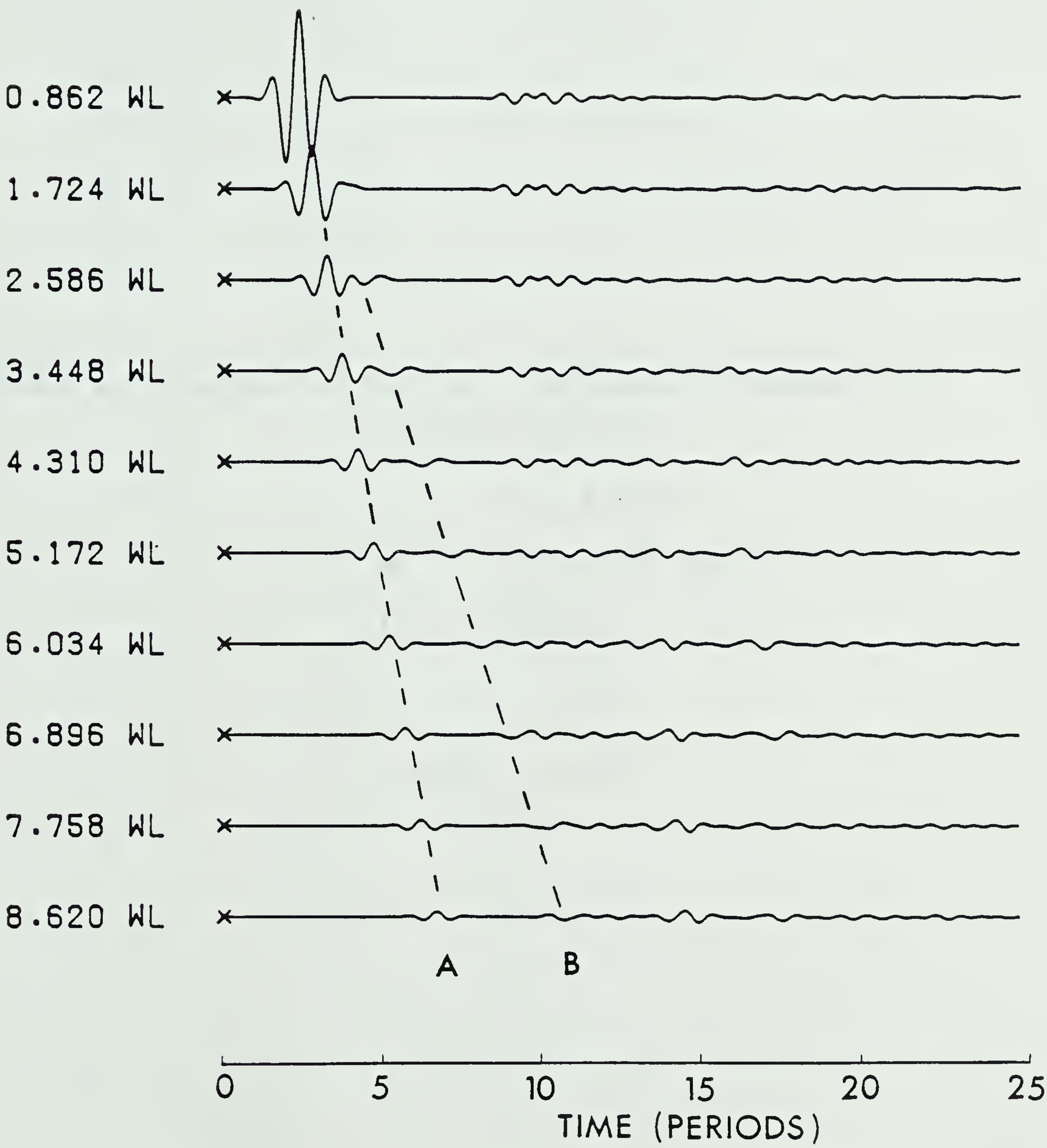

0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
A
B
C
D
0
5
10
15
20
25
TIME (PERIODS)

Figure 4.9 Explosive Source - Vertical Component of Surface Traces

A P_1

B Rayleigh Wave



0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
A
B
0
5
10
15
20
25
TIME (PERIODS)

Figure 4.10 Explosive Source - Horizontal Component of Vertical Profile

A	P_1, $\grave{P}_1\ S_2$
B	$\acute{P}_1\ S_1$, $\acute{P}_1\ S_1\ S_2$
C	$\grave{P}_1\ S_1$
D	$\acute{P}_1\ S_1{}^2$

Offset 4.31 WL

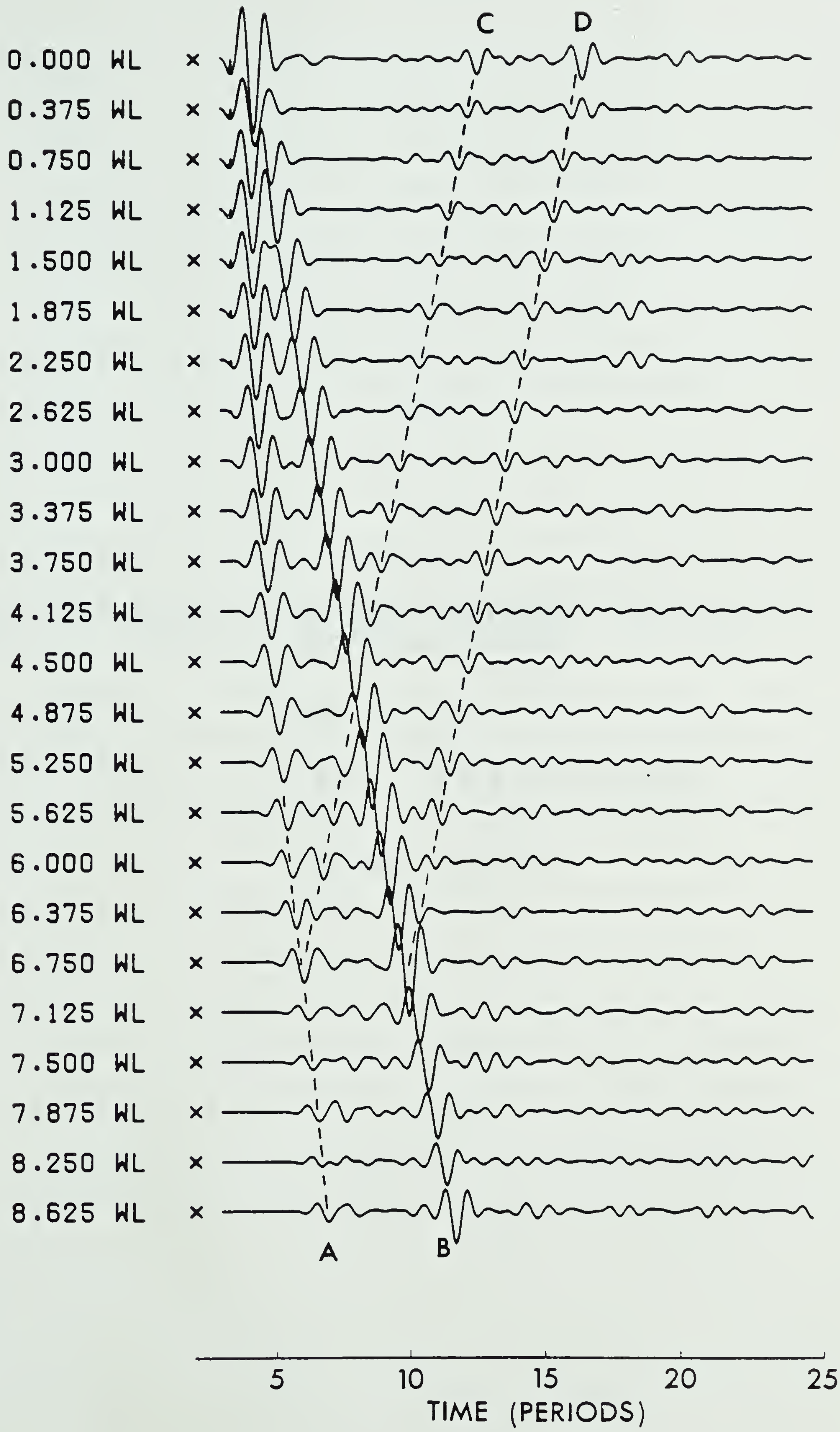

C
D
0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
A
B
5
10
15
20
25
TIME (PERIODS)

Figure 4.11 Explosive Source - Vertical Component of Vertical Profile

A - P_1

B $\acute{P}_1 S_1, \acute{P}_1 S_1 S_2$

Offset 4.31 WL

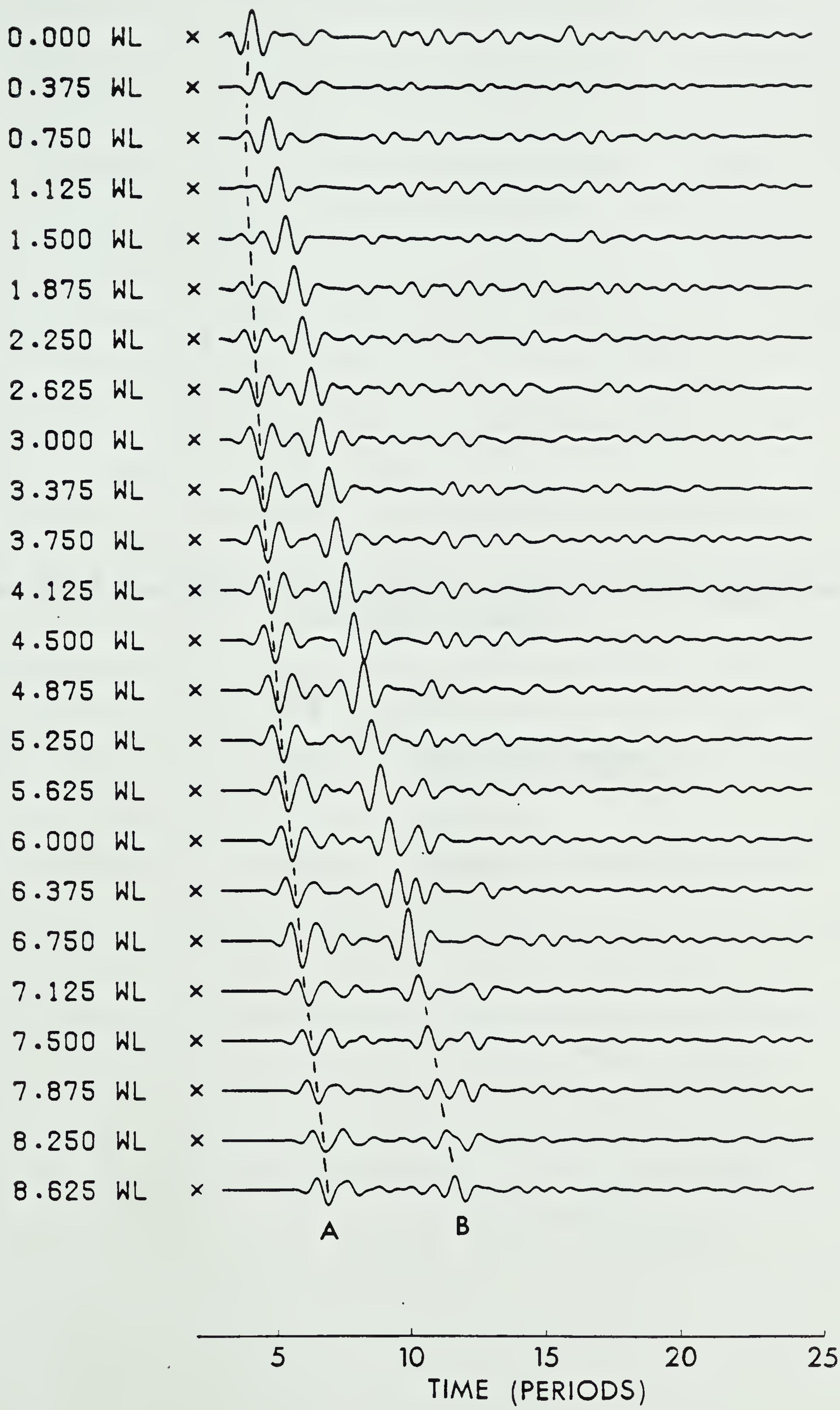

0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
A
B
5
10
15
20
25
TIME (PERIODS)

Figure 4.12 Vertical Point Force - Horizontal Component of Surface Traces

A	P_1
B	Rayleigh Wave
C	$\grave{P}_1$, S_1
D	$\grave{S}_1$, S_1

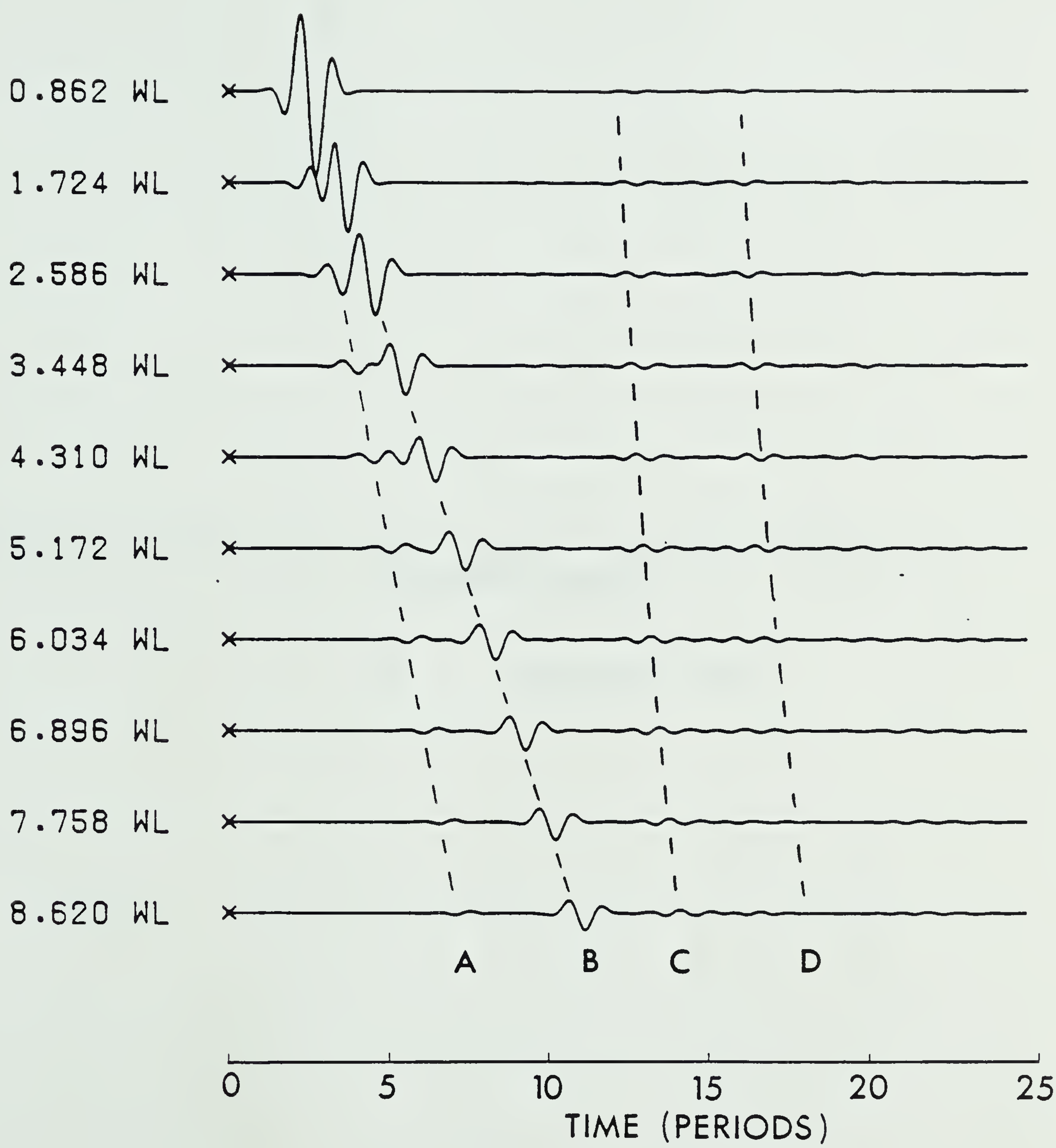
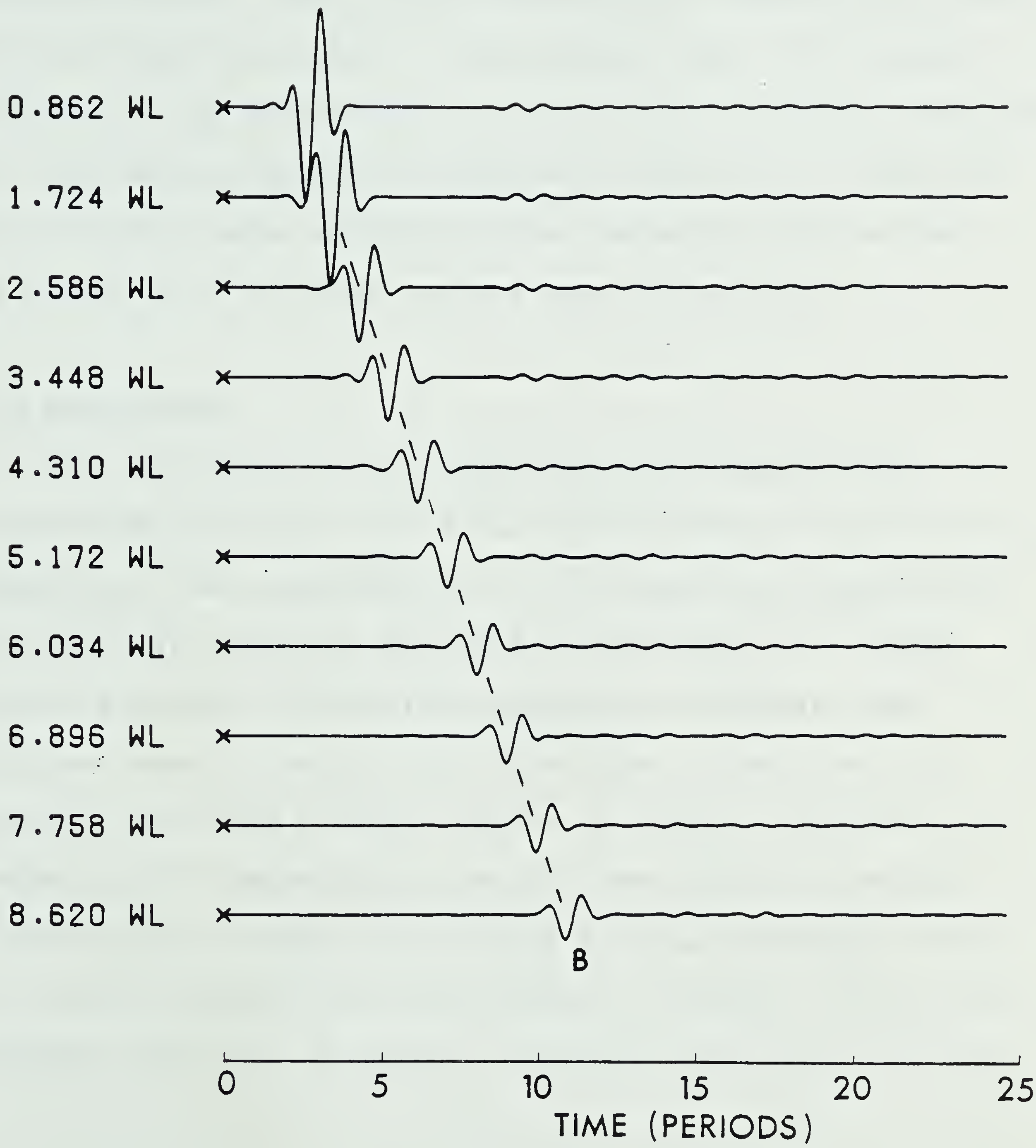

0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
A
B
C
D
0
5
10
15
20
25
TIME (PERIODS)

Figure 4.13 Vertical Point Force - Vertical Component of Surface Traces

B Rayleigh Wave


0.862 WL
1.724 WL
2.586 WL
3.448 WL
4.310 WL
5.172 WL
6.034 WL
6.896 WL
7.758 WL
8.620 WL
B
0
5
10
15
20
25
TIME (PERIODS)

4.7 Vertical Point Force - Vertical Profiles

Two features are immediately evident in Figures 4.14 and 4.15. The first is that there is both P and S radiation from the source. This is confirmed by the separation of the direct P and S arrivals on the surface trace. The second feature is the more complex nature of the wavefield produced by this source than by the explosive source as a result of both P and S waves emanating from the source and partial conversions at the free surface upon reflection.

4.8 Conclusion

For the generation of synthetic seismograms the Alekseev-Mikhailenko method has the advantage of providing a seemingly true representation of the wavefield produced by various point sources. Included in the traces are found direct arrivals, reflections and their multiples, and surface waves. An additional advantage is the numerical accuracy provided through the use of Hankel transforms in reducing the dimensionality of the wave equation thereby limiting grid dispersion. The use of this numerical method in vertical seismic profile studies is evident, however ray methods must still be used to identify particular arrivals.

Figure 4.14 Vertical Point Force - Horizontal Component of Vertical Profile

A P_1, P_1 P_2

B S_1, S_1 S_2

Offset 4.31 WL

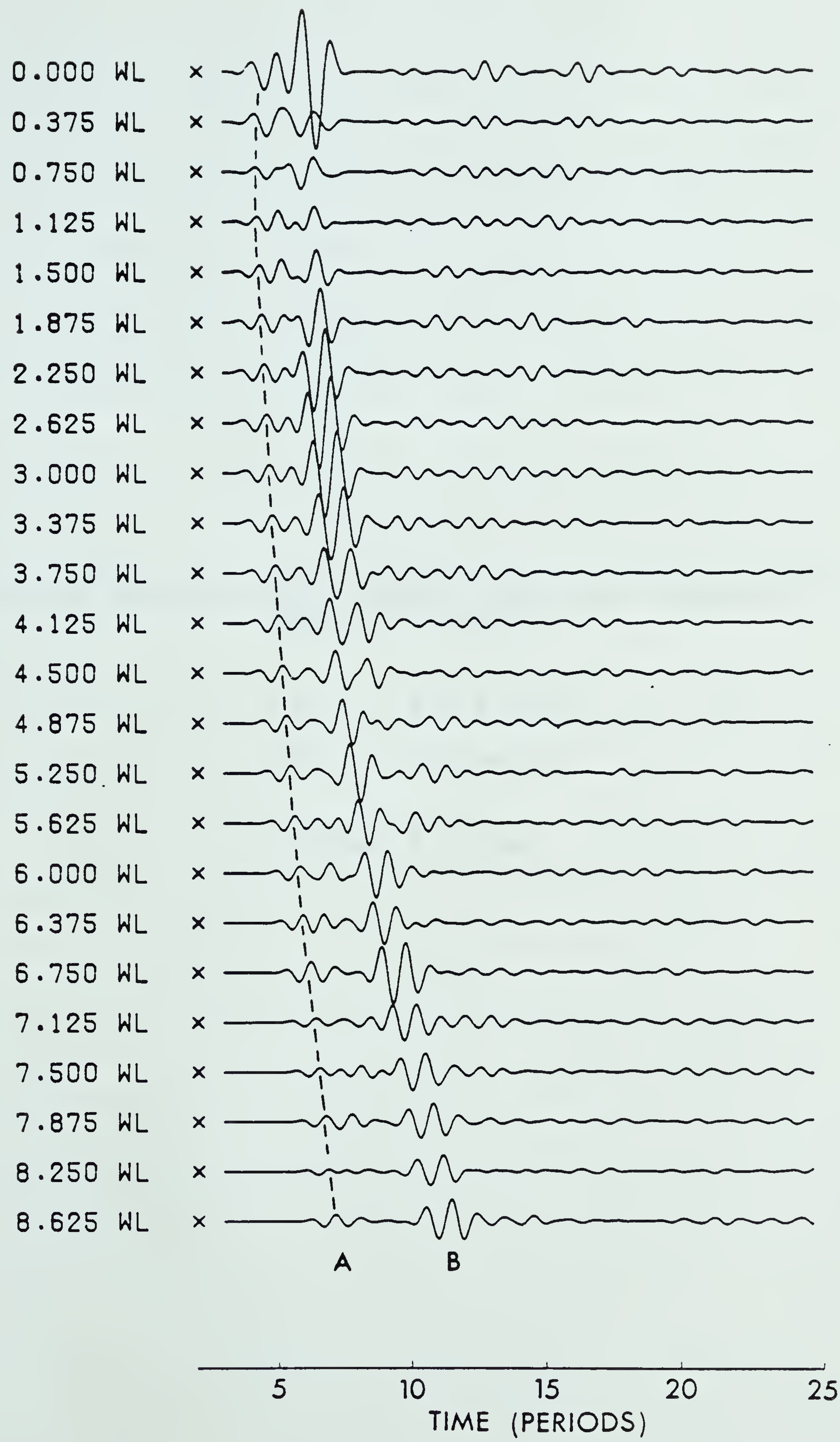
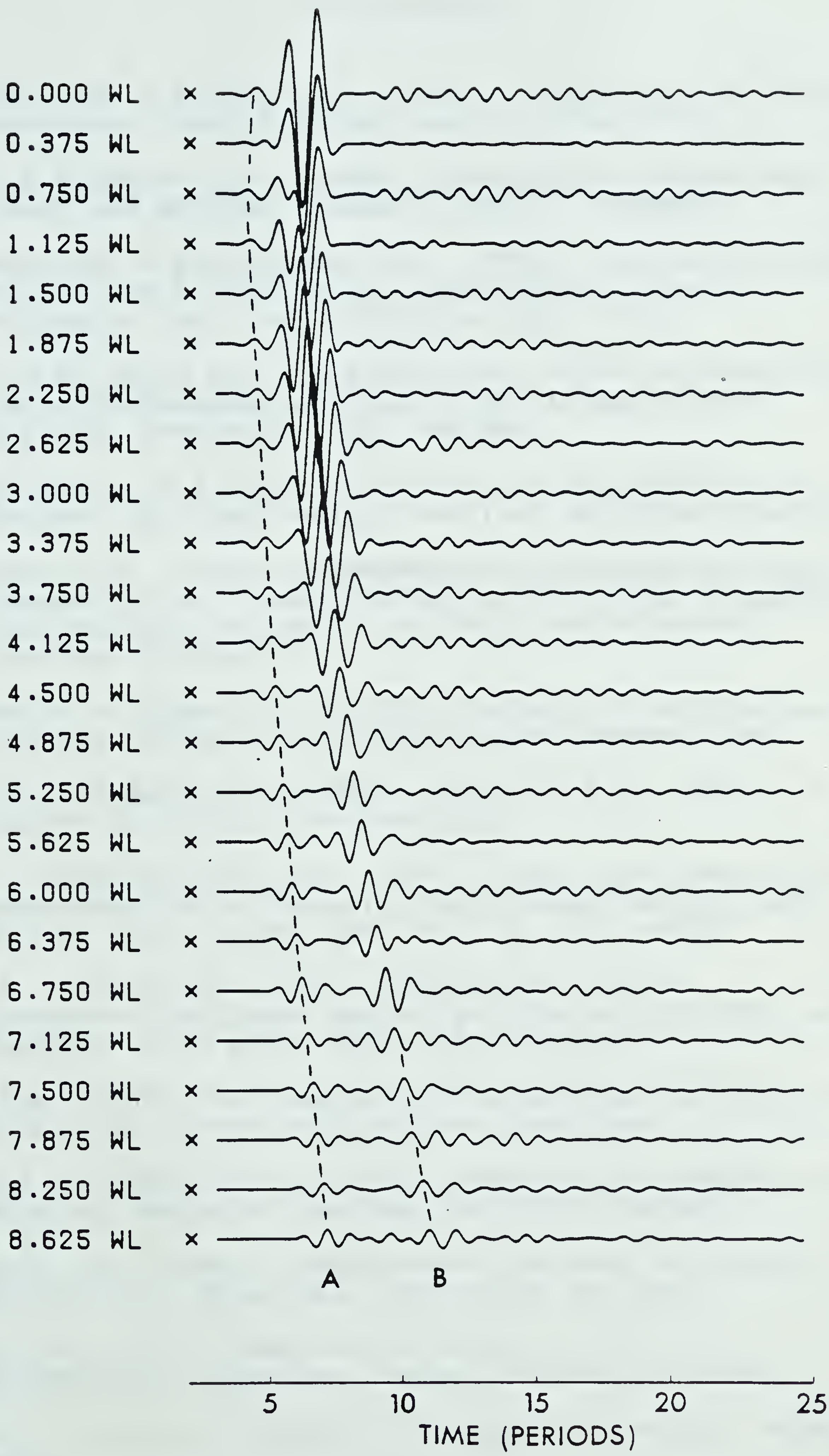

0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
A
B
5
10
15
20
25
TIME (PERIODS)

Figure 4.15 Vertical Point Force - Vertical Component of Vertical Profile

A P_1, $P_1 P_2$

B S_1, $S_1 S_2$

Offset 4.31 WL


0.000 WL
0.375 WL
0.750 WL
1.125 WL
1.500 WL
1.875 WL
2.250 WL
2.625 WL
3.000 WL
3.375 WL
3.750 WL
4.125 WL
4.500 WL
4.875 WL
5.250 WL
5.625 WL
6.000 WL
6.375 WL
6.750 WL
7.125 WL
7.500 WL
7.875 WL
8.250 WL
8.625 WL
A
B
5
10
15
20
25
TIME (PERIODS)

Bibliography

Abramowitz M. & Stegun I.A. (1968), Handbook of Mathematical Functions, Dover Publications Inc., New York.

Aki K. & Richards P.G. (1980), Quantitative Seismology: Theory and Methods Volumes I and II, Freeman.

Alekseev A.S. & Mikhailenko B.G. (1976), Solution of Lamb's Problem for a vertically inhomogeneous elastic halfspace, Izv., Earth Physics, 12, 11-25.

Alford R.M.,Kelly K.R., & Boore D.M. (1974), Accuracy of Finite-Difference Modeling of the Acoustic Wave Equation, Geophysics, 39, 834-842.

Ben-Menahem A. & Singh S.J. (1981), Seismic Waves and Sources, Springer-Verlag, New York Heidelberg Berlin.

Claerbout J.F. (1976), Fundamentals of Geophysical Data Processing: With Applications to Petroleum Prospecting, International Series in the Earth and Planetary Sciences, McGraw Hill.

Courant R. & Hilbert D., (1953), Methods of Mathematical Physics: Volume I, Interscience Publishers, Inc.

Grant F.S. & West G.F. (1965), Interpretation Theory in Applied Geophysics, McGraw-Hill.

Hron F. & Mikhailenko B.G. (1981), Numerical modelling of nongeometrical effects by the Alekseev-Mikhailenko method, Bull. Seism. Soc. Am., 71, 1011-1029.

Korn M. & Müller G. (1983), Comparison of the Alekseev-Mikhailenko method and the reflectivity method, Geophys. J. R. astr. Soc., 72, 541-556.

Love A.E. (1927), A Treatise On The Mathematical Theory of Elasticity, Dover Publications, New York.

Mathews J. & Walker R.L. (1970), Mathematical Methods of Physics ,Benjamin Cummings Publishing Company.

Mitchell A.R. (1969), Computational Methods in Partial Differential Equations, John Wiley and Sons.

Smith G.D. (1965), Numerical Solution of Partial Differential Equations, Oxford University Press.

Sneddon I. (1972), The Use of Integral Transforms, McGraw Hill.